The Slice Tower and Suspensions

Carolyn M. Yarnall

Wabash College

yarnallc@wabash.edu

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Setup

Background

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- A key part of the solution to the Kervaire Invariant-One problem.

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Basic Idea

- The construction of the slice tower is analogous to that of the Postnikov tower.
- Instead of killing maps from spheres, we kill maps from *slice cells*.

Basic Definitions

Notation

- G is a finite group.
- X is a G-spectrum.
- ρ_G is the regular representation of G.
- S^V is the 1-point compactification of a representation space V.

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Let $\tau_{\geq n}$ denote the localizing subcategory of G-spectra generated by slice cells of dimension $\geq n$.

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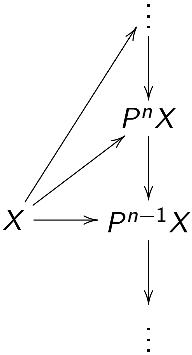

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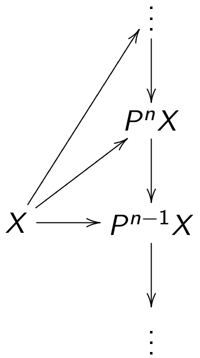

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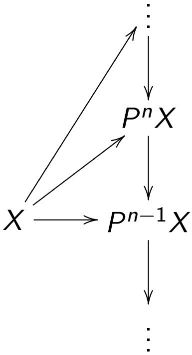
- Then for any G-spectrum X we have a tower.
- Its limit is X and its colimit is contractible.



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$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ P_{n+1}^{n+1}X & \longrightarrow & P^{n+1}X \\ & & \downarrow \\ P_n^n X & \longrightarrow & P^n X \\ & & \downarrow \\ P_{n-1}^{n-1}X & \longrightarrow & P^{n-1}X \\ & & \downarrow \\ & & \vdots \end{array}$$

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- $P_n^n(X) \geq n$

That is, $P_n^n X \in \tau_{\geq n}$.

- $P_n^n(X) \leq n$

That is, $P_n^n X \to P^{n-1}(P_n^n X)$ is an equivalence.

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ P_{n+1}^{n+1}X & \longrightarrow & P^{n+1}X \\ & & \downarrow \\ P_n^n X & \longrightarrow & P^n X \\ & & \downarrow \\ P_{n-1}^{n-1}X & \longrightarrow & P^{n-1}X \\ & & \downarrow \\ & & \vdots \end{array}$$

An Example: $P^{-1}_{-1}(X)$

$$\begin{aligned} \tau_{\geq 0} &= \langle G/H_+ \rangle \\ &= \{(-1)\text{-connected } G\text{-spectra}\} \end{aligned}$$

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$P^{-1}X$	$P^{-2}X$
0	0
0	0
$\pi_{-1}(X)$	0
$\pi_{-2}(X)$	$\pi_{-2}(X)$
$\pi_{-3}(X)$	$\pi_{-3}(X)$
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$\vdots$

$P^{-2}X$
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$$P^{-1}_{-1}X \to P^{-1}X \to P^{-2}X$$

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$$P^{-1}_{-1}X \to P^{-1}X \to P^{-2}X$$

All (-1)-slices can be given as:

$$\boxed{P^{-1}_{-1}(X) = \Sigma^{-1} H\underline{\pi_{-1}(X)}}$$

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- The slice tower is not necessarily trivial for E-M spectra.
- More generally, in constructing $P^n X$ for $n \geq 0$, lower homotopy groups may be affected so the slices are not necessarily E-M spectra.

Suspensions and the Slice Tower

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Theorem [Hill-Hopkins-Ravenel]

The slice tower commutes with suspensions by regular representations:

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and hence,

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and hence,

$$P_{k+m|G|}^{k+m|G|}(\Sigma^{m\rho_G} X) = \Sigma^{m\rho_G} P_k^k X$$

Result

All $(m|G|-1)$-slices of X are:

$$P_{m|G|-1}^{m|G|-1} X = \Sigma^{m\rho_G} P_{-1}^{-1}(\Sigma^{-m\rho_G} X) = \Sigma^{m\rho_G - 1} H\underline{\pi_{-1}(\Sigma^{-m\rho_G} X)}$$

Determining Slices for $S^n \wedge H\underline{\mathbb{Z}}$

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Compute the slice towers for $X = S^n \wedge H\underline{\mathbb{Z}}$ where $G = C_{p^k}$.

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We need a tower satisfying appropriate properties (limit, colimit, fibers are slices).

To Do:

- What dimensions are the nontrivial slices in?
- What do they look like?
- What fiber sequences do they fit into?

Some Slices of $S^n \wedge H\underline{\mathbb{Z}}$

We get the $(mp^k - 1)$-slices as:

$$P^{mp^k-1}_{mp^k-1}X = \Sigma^{m\rho_G - 1} H\underline{\pi_{-1}(S^{n-m\rho_G} \wedge H\underline{\mathbb{Z}})}$$

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Using chain complexes of Mackey functors we compute:

$$\underline{H_{-1}(S^{n-m\rho_G}; \underline{\mathbb{Z}})} = \underline{\pi_{-1}(S^{n-m\rho_G} \wedge H\underline{\mathbb{Z}})}$$

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Theorem 1 [Y]

Let $G = C_{p^k}$ for p an odd prime.

$$P^{mp^k-1}_{mp^k-1}(S^n \wedge H\underline{\mathbb{Z}}) = \begin{cases} \Sigma^{m\rho_G - 1} H\underline{B}_{(k,j)} & m, n \text{ of same parity} \\ * & \text{otherwise} \end{cases}$$

Remaining Slices of $S^n \wedge H\underline{\mathbb{Z}}$

Theorem 2 [Y.]

The nontrivial slices of $S^n \wedge H\underline{\mathbb{Z}}$ where $G = C_{p^k}$ are:

- only in dimensions n and $(mp^a - 1)$ where $1 \leq a \leq k$ and m is as in Theorem 1.
- of the form $S^{V_a} \wedge H\underline{B}_{(\nu_p(m)+a,a-1)}$ where

$$V_a = (n-2)\rho_G - 1 - \bigoplus_{i=1}^{L} \lambda(i)$$

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Fiber Sequences

Lemma

There are fiber sequences of the form

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Suspending by V_a gives the sequences to be used in the tower:

$$\begin{array}{ccc} S^{V_a} \wedge H\underline{B}_{(\nu_p(m)+a,a-1)} & \longrightarrow & S^{V_a+1+\lambda(p^{\nu_p(m)+a})} \wedge H\underline{\mathbb{Z}} \\ & & \downarrow \\ & & S^{V_a+1+\lambda(p^{a-1})} \wedge H\underline{\mathbb{Z}} \end{array}$$

Example: $n = 7$, $p = 3$, $k = 2$

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The slice tower for $S^7 \wedge H\underline{\mathbb{Z}}$ where $G = C_9$ is as follows:

$$
\begin{array}{lccc}
(5p^2-1)\text{-slice:} & S^{5\rho-1} \wedge H\underline{B}_{(2,1)} & \longrightarrow & S(7,0,0) \\
 & & & \downarrow \\
(3p^2-1)\text{-slice:} & S^{3\rho-1} \wedge H\underline{B}_{(2,1)} & \longrightarrow & S(5,1,0) \\
 & & & \downarrow \\
(5p-1)\text{-slice:} & S^{2+\lambda(3)} \wedge H\underline{B}_{(1,0)} & \longrightarrow & S(3,2,0) \\
 & & & \downarrow \\
(3p-1)\text{-slice:} & S^{\rho-1} \wedge H\underline{B}_{(2,0)} & \longrightarrow & S(3,1,1) \\
 & & & \downarrow \\
 & & & S(1,1,2)
\end{array}
$$

Comparative Examples

$$\begin{array}{ccc}
S^{5\rho-1} \wedge H\underline{B}_{(2,1)} & \longrightarrow & S(7,0,0) \\
 & & \downarrow \\
S^{3\rho-1} \wedge H\underline{B}_{(2,1)} & \longrightarrow & S(5,1,0) \\
 & & \downarrow \\
S^{2+\lambda(3)} \wedge H\underline{B}_{(1,0)} & \longrightarrow & S(3,2,0) \\
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\end{array}
$$

$$
\begin{array}{ccc}
S^{14\rho-1} \wedge H\underline{B}_{(2,1)} & \to & S(16,0,0) \\
 & & \downarrow \\
S^{12\rho-1} \wedge H\underline{B}_{(2,1)} & \to & S(14,1,0) \\
 & & \downarrow \\
S^{10\rho-1} \wedge H\underline{B}_{(2,1)} & \to & S(12,2,0) \\
 & & \downarrow \\
S^{8\rho-1} \wedge H\underline{B}_{(2,1)} & \to & S(10,3,0) \\
 & & \downarrow \\
S^{6\rho-1} \wedge H\underline{B}_{(2,1)} & \to & S(8,4,0) \\
 & & \downarrow \\
S^{5+4\lambda(3)} \wedge H\underline{B}_{(1,0)} & \to & S(6,5,0) \\
 & & \downarrow \\
S^{4\rho-1} \wedge H\underline{B}_{(2,0)} & \to & S(6,4,1) \\
 & & \downarrow \\
S^{3+3\lambda(3)} \wedge H\underline{B}_{(1,0)} & \to & S(4,4,2) \\
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S^{3+2\lambda(3)} \wedge H\underline{B}_{(1,0)} & \to & S(4,3,3) \\
 & & \downarrow \\
S^{2\rho-1} \wedge H\underline{B}_{(2,0)} & \to & S(4,2,4) \\
 & & \downarrow \\
 & & S(2,2,5)
\end{array}
$$

The Slices of $S^{16} \wedge H\underline{\mathbb{Z}}$ for $G = C_{p^3}$ with $p = 3$

$(mp^3 - 1)$-slices

$S^{14\rho-1} \wedge H\underline{B}_{(3,2)}$

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$(mp^2 - 1)$-slices

$S^{5+4\lambda(3)} \wedge H\underline{B}_{(2,1)}$

$S^{4\rho-1} \wedge H\underline{B}_{(3,1)}$

$S^{3+3\lambda(3)} \wedge H\underline{B}_{(2,1)}$

$S^{3+2\lambda(3)} \wedge H\underline{B}_{(2,1)}$

$S^{2\rho-1} \wedge H\underline{B}_{(3,1)}$

$(mp - 1)$-slices

$S^{1+2\lambda(3)+4\lambda(1)} \wedge H\underline{B}_{(1,0)}$

$S^{1+\lambda(3)+4\lambda(1)} \wedge H\underline{B}_{(2,0)}$

$S^{1+\lambda(3)+3\lambda(1)} \wedge H\underline{B}_{(1,0)}$

$S^{1+\lambda(3)+2\lambda(1)} \wedge H\underline{B}_{(1,0)}$

$S^{1+2\lambda(1)} \wedge H\underline{B}_{(2,0)}$