Towards a Resolution of the $K(2)$-local Sphere at the Prime 2.

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Homotopy Groups of Spheres

Consider the sphere spectrum S.

Question

How do we compute $\pi_* S$?

Answer

We choose appropriate localizations so that the problem becomes approachable.

Chromatic Homotopy Theory

- Fix a prime p.
- The Johnson-Wilson theories $\{E(n)\}_{n=0,1,\ldots}$ allow to filter the category of p-local spectra.
- Localizations with respect to Johnson-Wilson theories form chromatic tower

$$\ldots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X.$$

Chromatic Convergence Theorem (Hopkins, Ravenel)

For a finite p-local spectrum X

$$X = \operatorname{holim}_n L_{E(n)}X.$$

Morava K-theory

Let $K(n)$ denote n-th Morava K-theory.

Theorem (Ravenel, Hovey-Strickland)

There is a homotopy pullback diagram:

$$\begin{array}{ccc} L_{E(n)}X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)}L_{K(n)}X. \end{array}$$

So we can concentrate on computing $\pi_* L_{K(n)} S$. We do it with the help of Morava E-theory and Morava Stabilizer Group.

Morava E-theory

Theorem (Morava, Goerss-Hopkins-Miller, Devinatz)

- *For each n there exists a spectrum E_n, called the n-th Morava E-theory,*
- *and a group $\mathbb{G}_n$, called the Morava Stabilizer group.*
- *$\mathbb{G}_n$ acts on E_n.*
- *for H a closed subgroup of $\mathbb{G}_n$ we can form homotopy fixed points spectra E_n^{hH}.*
- *$E_n^{h\mathbb{G}_n} = L_{K(n)}S^0$.*
- *For any closed subgroup H of $\mathbb{G}_n$ there is a spectral sequence*

$$E_2^{s,t} = H^*(H, (E_n)_*) \Longrightarrow \pi_* E_n^{hH}.$$

Known results, n=1, p=2

Theorem (Adams, Baird, Ravenel)

For $n = 1$ and $p = 2$ there is the fiber sequence

$$L_{K(1)}S^0 \to KO\mathbb{Z}_2 \to KO\mathbb{Z}_2,$$

which is equivalent to

$$E_1^{h\mathbb{G}_1} \to E_1^{hC_2} \to E_1^{hC_2}.$$

Known results, n=2

Using the spectral sequence

$$E_2^{*,*} = H_c^*(\mathbb{G}_2, (E_2)_*) \Longrightarrow \pi_* L_{K(2)} S^0 :$$

- At $p \geq 5$ Shimomura and Yabe computed $\pi_* L_{K(2)} S^0$.
- At $p = 3$ $\mathbb{G}_2$ contains C_3
  Shimomura and Wang computed $\pi_* L_{K(2)} S^0$.
- At $p = 2$ $\mathbb{G}_2$ contains Q_8
  Shimomura and Wang computed the second page of the spectral sequence.

Different approach

Plan

Try to build the $K(2)$-local sphere spectrum out of $E_2^{hH_i}$ for finite subgroups H_i of $\mathbb{G}_2$.

Work with the subgroup $\mathbb{G}_2^1$ of $\mathbb{G}_2$, such that there is a fiber sequence

$$L_{K(2)}S^0 \to E^{h\mathbb{G}_2^1} \to E^{h\mathbb{G}_2^1}.$$

Known results, n=2, p=3

Theorem (Goerss, Henn, Mahowald, Rezk)

There exists a resolution in the $K(2)$-local category at the prime 3

$$E^{h\mathbb{G}_2^1} \to E^{hG_{24}} \to \Sigma^8 E^{hSD_{16}} \to \Sigma^{40} E^{hSD_{16}} \to \Sigma^{48} E^{hG_{24}}$$

which can be realized to a tower of fibrations:

$$\begin{array}{ccc}
\Sigma^{45} E^{hG_{24}} & \longrightarrow & E^{h\mathbb{G}_2^1} \\
 & & \downarrow \\
\Sigma^{38} E^{hSD_{16}} & \longrightarrow & X_2 \\
 & & \downarrow \\
\Sigma^{7} E^{hSD_{16}} & \longrightarrow & X_1 \\
 & & \downarrow \\
E^{hG_{24}} & \longrightarrow & E^{hG_{24}}
\end{array}$$

Tower Spectral Sequence

Given a tower of fibrations with limit X and fibers F_i

$$\begin{array}{ccccccc} Z & \longrightarrow & X_n & \longrightarrow & \dots & \longrightarrow & X_0 \\ \uparrow & & \uparrow & & & & \uparrow \\ F_{n+1} & & F_n & & & & F_0 \end{array}$$

there exists a spectral sequence

$$E_1^{s,t} = \pi_{t-s} F_s \Longrightarrow \pi_{t-s} Z.$$

Conjecture (Mahowald, Rezk, Behrens)

For $p > 2$ there exist spectra $Q(N)$ such there is a fiber sequence

$$DQ(N) \to L_{K(2)}S \to Q(N).$$

Theorem (Behrens)

The conjecture is true for $p = 3$.

Proof uses the GHMR resolution.

Known results, n=2, p=2

Theorem (Goerss, Henn, Mahowald, Rezk)

There exists a resolution in the $K(2)$-local category at the prime 2

$$E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \to E^{hC_6} \to E^{hC_6} \to X$$

which can be realized to a tower of fibrations:

$$\begin{array}{ccc}
\Sigma^{-3}X & \longrightarrow & E^{h\mathbb{S}_2^1} \\
 & & \downarrow \\
\Sigma^{-2}E^{hC_6} & \longrightarrow & X_2 \\
 & & \downarrow \\
\Sigma^{-1}E^{hC_6} & \longrightarrow & X_1 \\
 & & \downarrow \\
E^{hG_{24}} & \longrightarrow & E^{hG_{24}}
\end{array}$$

New results, n=2, p=2

Theorem (B.)

In the tower of fibrations:

$$\begin{array}{ccc}
\Sigma^{-3}X & \longrightarrow & E^{h\mathbb{S}_2^1} \\
 & & \downarrow \\
\Sigma^{-2}E^{hC_6} & \longrightarrow & X_2 \\
 & & \downarrow \\
\Sigma^{-1}E^{hC_6} & \longrightarrow & X_1 \\
 & & \downarrow \\
E^{hG_{24}} & \longrightarrow & E^{hG_{24}}
\end{array}$$

$\pi_* X = \pi_* \Sigma^{48} E^{hG_{24}}$.

Idea of the proof

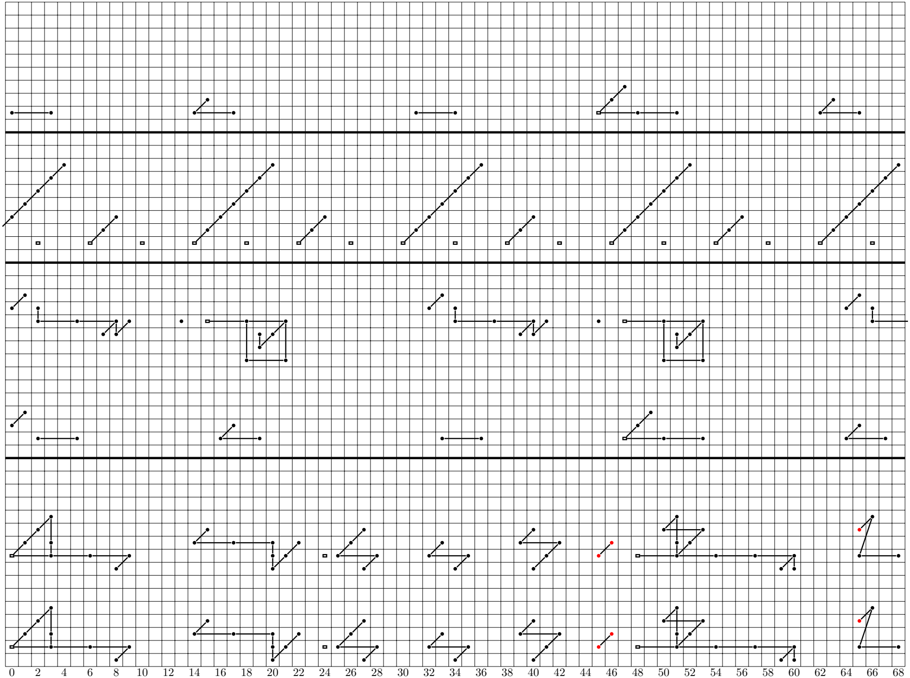
Theorem (Henn)

There exists a resolution in the $K(2)$-local category

$$E^{h\mathbb{S}_2^1} \to E^{hG_{24}} \vee E^{hG_{24}} \to E^{hC_6} \vee E^{hC_4} \to E^{hC_2} \to E^{hC_6}$$

which can be realized to a tower of fibrations:

$$\begin{array}{ccc}
\Sigma^{-3} E^{hC_6} & \longrightarrow & E^{h\mathbb{S}_2^1} \\
 & & \downarrow \\
\Sigma^{-2} E^{hC_2} & \longrightarrow & X_2 \\
 & & \downarrow \\
\Sigma^{-1}(E^{hC_6} \vee E^{hC_4}) & \longrightarrow & X_1 \\
 & & \downarrow \\
E^{hG_{24}} \vee E^{hG_{24}} & \longrightarrow & E^{hG_{24}}
\end{array}$$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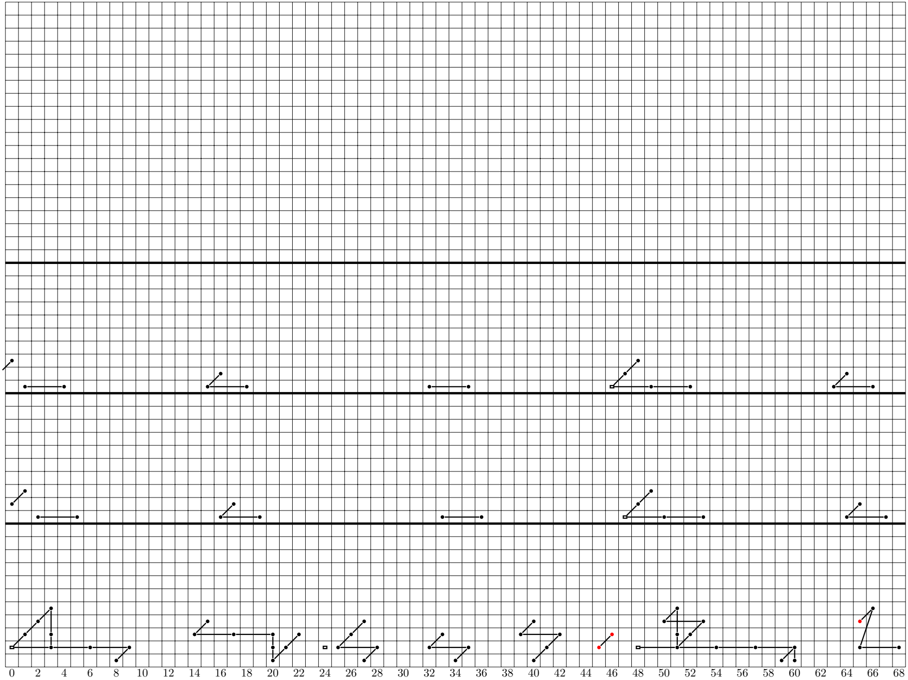
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New Results, n=2, p=2

Lemma

Δ^{2+8i} is a homotopy class in $\pi_* X$.

Theorem (Folklore, Hopkins-Mahowald)

If Δ^{2+8i} is a homotopy class in $\pi_* X$, then $\pi_* X = \pi_* \Sigma^{48} E^{hG_{24}}$.

Work in progress

Conjecture

X is homotopy equivalent to $\Sigma^{48} E^{hG_{24}}$.