What should π_0 be ?

I : Introductory examples

II : Compact transformation groups

III : Proper Lie groupoids

IV : Closing examples

Thanks to the organizers!!

§I : Introductory examples

Homotopy theory is arguably the study of the derived functors of π_0, and I want to suggest here that in some interesting (overlapping) cases, in

- **geometric** topology (ie **not** the study of simplicial sets!),

- higher category theory,

- big data, and

- (possibly) in homotopy **type theory**

π_0 merits a closer look. I will try to illustrate this with examples from the theory of group actions and their generalizations.

Definition: A topological group action $G \times X \to X$ has an associated topological groupoid

$$[X/G] \;:=\; s, t : G \times X \rightrightarrows X\,.$$

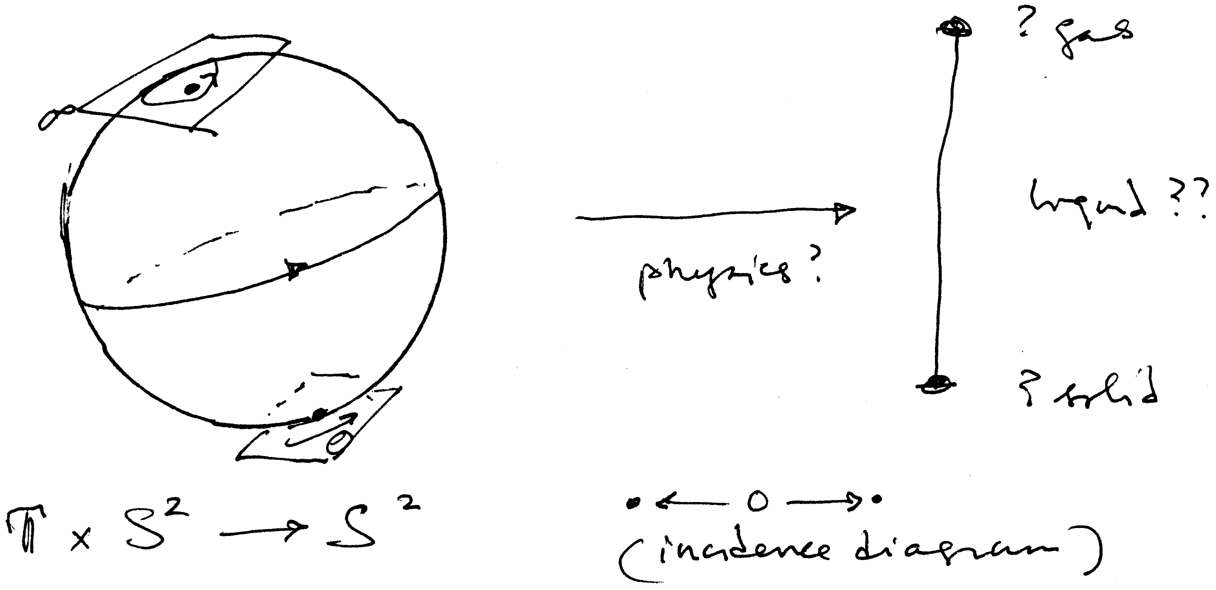
(or **transformation group**).

The quotient space

$$[X/G] \to X/G$$

can in general be very bad, but it often has a useful decomposition or **stratification** into nice pieces, usually called orbit types.

Example: [cf Duistermaat & Kolk, **Lie groups** II §2.8]



$$G = \{u \in \mathbb{C} \mid |u| = 1\} = \mathbb{T}$$

$$X = \mathbb{C} \cup \infty = S^2 = \mathbb{C}P^1$$

$$(u, z) \mapsto u \cdot z : \mathbb{T} \times S^2 \to S^2 \ .$$

This generalizes to **toric varieties**: thus

$$G = \mathbb{T}^{n+1}/\mathbb{T} \ni (u_0 : \cdots : u_n) = \mathbf{u}$$

$$X = \mathbb{C}P^n \ni [z_0 : \cdots : z_n] = \mathbf{z}$$

$$(\mathbf{u}, \mathbf{z}) \mapsto \mathbf{u} \cdot \mathbf{z} = [u_0 z_0 : \cdot : u_n z_n] : G \times X \to X \ ,$$

has the n-simplex $\Delta^n = X/G$ as quotient, and the category of subsets of $\{0, \ldots, n\}$ as its stratification poset.

[More general polytopes occur, eg for suitable

$$\mathbf{u}, \mathbf{z} \mapsto [u_0^{i_0} z_0 : \cdot : u_n^{i_n} z_n], \ i_0, \ldots, i_n \geq 1]$$

Remarks:

• Algebraic varieties have a canonical stratification (eg of their real or complex points [Whitney, Lojasiewicz, . . .]).

• The Zariski spectrum of a Noetherian ring is a (non-Hausdorff) **Noetherian space**, with a stratification by codimension.

• Stratifications are very important in physics (eg in symplectic geometry and geometric quantization), in understanding symmetry breaking and phase transitions.

§II Compact transformation groups

To be more specific, suppose G is a locally Euclidean compact topological group. If $H < G$ is a closed subgroup, then

$$X^H = \{x \in X \mid h \in H \Rightarrow hx = x\} \subset X$$

is the closed subspace of H-fixed points in X, ie with $H \subset \mathrm{Iso}(x)$. Thus

$$X^H = \overline{X_H} \supset X_H = \{x \in X \mid \mathrm{Iso}(x) = H\} .$$

This is the source of the orbit type stratification of X (or X/G).

Claim: There is a presheaf

$$X^\bullet : H \mapsto X^H$$

of spaces on the (topological, two-) category of G-orbits.

Here $(G - \mathrm{Orb})$ has closed subgroups of G as its objects, with the transformation group

$$[\{\phi \in \mathrm{Hom}_c(H_0, H_1) | \exists g \in G,\ \phi(h) = ghg^{-1}\}/H_1^{\mathrm{conj}}]$$

(whose quotient space is $G-\mathrm{Maps}(G/H_0, G/H_1)$) as the morphism category

$$\mathrm{Hom}_{\mathrm{Orb}}(H_0, H_1) .$$

Such constructions lie behind the standard approach to equivariant homotopy theory, which analyzes a G-space (or spectrum) in terms of this sheaf of fixed-point objects.

Ad Hoc **Definition:** The fiber product category

$$\begin{array}{ccc} \Phi_0[X/G] & \dashrightarrow & (\mathsf{Sets})_* \\ \downarrow & & \downarrow \\ \pi_0 X^\bullet : (G - \mathsf{Orb}) & \longrightarrow & (\mathsf{Sets}) \end{array}$$

(with objects, pairs (H, C) with $H < G$ and C a component of X^H) admits a natural transformation

$$x \mapsto (\mathsf{Iso}(x), [x] \in \pi_0(X^{\mathsf{Iso}(x)})) :$$

$$[X/G] \to \Phi_0[X/G]$$

which might be regarded as a subtler version of the classical map

$$X/G \to \pi_0(X/G) \, .$$

Remarks:

• I propose to think of $\Phi_0[X/G]$ as a **database** (in the sense of D Spivak: roughly, as a finitely presented category) of orbit types.

• I don't know how to characterize this construction in terms of universal properties, and (since it depends on the presentation of $[X/G]$ in terms of a global group action) it doesn't look very homotopy-invariant.

§III Proper Lie groupoids

A groupoid

$$\mathbf{X} := s, t : X_1 \rightrightarrows X_0 \, .$$

in the category of smooth manifolds is a proper Lie groupoid (PLG for short) if the product map

$$s \times t : X_1 \to X_0 \times X_0$$

is proper. Its isotropy groups are consequently compact Lie; orbifolds form a very special class of examples.

Claim: If H is a compact Lie group, there is a groupoid $\mathbf{X}(H)$ of fixed-points **of level** H, with a space $X(H)_0 =$

$$\{(x, \phi) \mid x \in X_0, \; \phi : H \rightarrowtail \mathsf{Iso}(x)\}$$

of objects (and morphisms defined by conjugation).

Because $\mathbf{X}$ is not presented by a global group action, $\mathbf{X}(H)$ is more complicated than X^H. For example, there is a map

$$[X/G](H)_0 \cong \coprod_{H<K<G} X_K \times \mathsf{Hom}_c^{\mathsf{inj}}(H,K)$$

$$\cdots \to X^H = \overline{X_H} = \coprod_{H<K<G} X_K$$

which defines a map from the quotient space of $[X/G](H)$ to X^H, with the spaces

$$\mathsf{Hom}_c^{\mathsf{inj}}(H,K)/K^{\mathsf{conj}}$$

(of identifications of the isotropy groups with H) as fibers.

Recall that for transformation groups $[X/G]$, a **slice** at $x \in X$ is an analog of an exponential map germ

$$G \times_{\text{Iso}(x)} \mathcal{N}_x \xrightarrow{\cong} \text{nbd of } x \in X$$

defined by a presentation of the normal bundle to the orbit of x in terms of a linear representation $\mathcal{N}_x$ of Iso(x).

The fixed point groupoids $\mathbf{X}(H)$ are (arguably) accessible because of a recent (`arXiv:1101.0180`)

Theorem of Pflaum *et al*:

There is a good theory of slices for PLGs!

Corollary: Any PLG $\mathbf{X}$ has a canonical **Whitney** stratification.

[This is defined by components of the equivalence relation $x \sim y \in X_0$ iff $\mathcal{N}_x$ (as an Iso(x)-representation) is isomorphic to $\mathcal{N}_y$ (as an Iso(y)-representation).]

Claim:

$$\mathbf{X} \mapsto [H \mapsto \pi_0 \mathbf{X}(H)]$$

leads to a generalization of Φ_0 from transformation groups to PLGs.

Moreover, there is a naturally associated fibered category

$$\mathcal{N}(\mathbf{X}) \to \Phi_0(\mathbf{X})$$

of normal slice representations.

Remarks:

• A classical theorem of Noether associates conserved physical quantities ('order parameters') to elements of the Lie algebra of symmetries of states of certain physical systems. The category $\mathcal{N}(\mathbf{X})$ looks like a useful repository for such information.

• The remarks in §**III**, about homotopy invariance and universal properties, continue to apply here.

§IV: Concluding examples:

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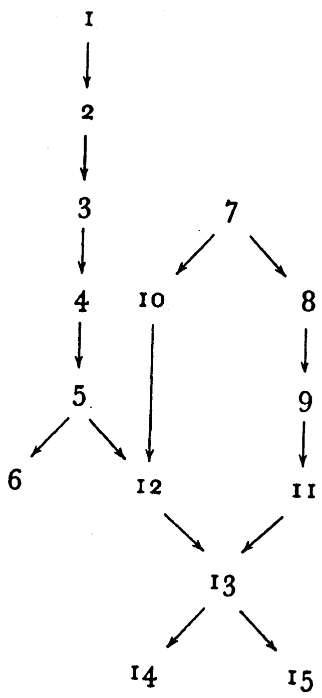
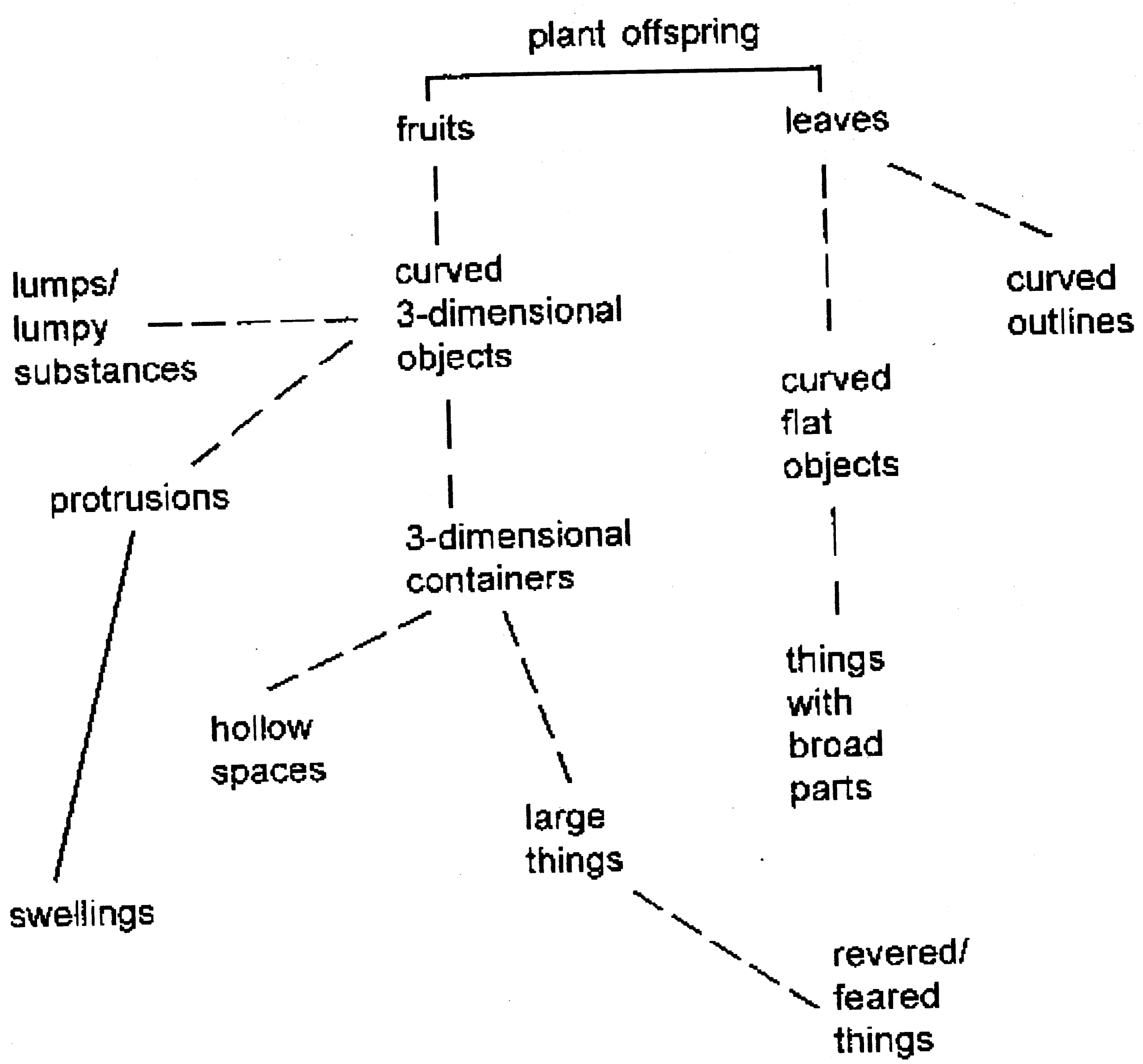


Organizational scheme for JP Serre, **Corps locaux**, Hermann (1962)

A semantic network for Class 5.

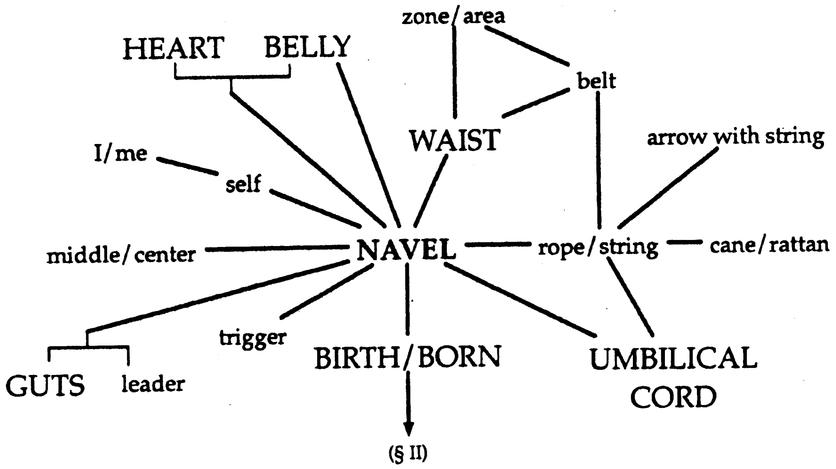


From Ellen Contini-Morava, **Noun classification in Swahili** (§4.3),

http://www2.iath.virginia.edu/swahili/swahili.html

[A stratification of the nouns in gender classes 5 and 9, traditionally understood as having something to do with plants]

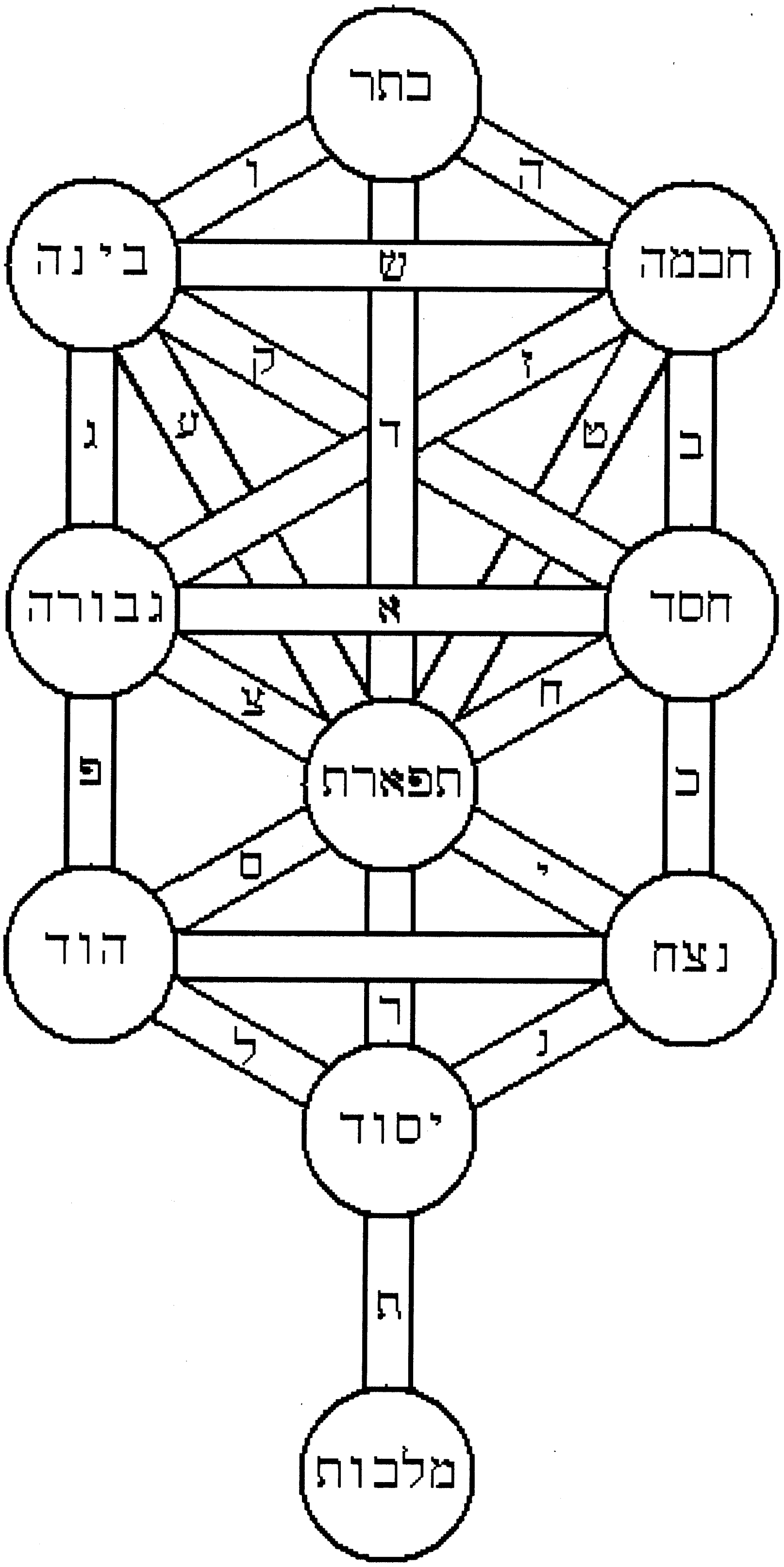
III. Navel



(40) *$^{m}_{s}$-la(:)y ⪤ *s-tay NAVEL

A root ***la:y** is set up in *STC* #287 with the meaning MIDDLE/CENTER, on the basis of WB **ălai** and Lushai **lai.**[1] Elsewhere, *STC* presents two additional roots: ***s-tay** 'navel' (*STC* #299), based on WT **lte-ba**, Jingpho **dài ~ šədăi**, Garo **ste**; and ***tay** 'self' (*STC* #284), based on Jingpho **dăi** (also **dăidăi**) and Lushai **tei**. Yet Benedict himself implies by a cross-reference (p. 65) that these latter two roots are really one and the same. I wish to go a step further to claim that all three *STC* roots (#284, #287, #299) are co-allofamic.

From J Matisoff, **The Tibeto-Burman reproductive system**: towards an etymological thesaurus. U of Cal Pub in Linguistics # 139



Some metaphysical categories in Medieval Jewish philosophy

http : //en.wikipedia.org/wiki/Sephirot

CATEGORIES OF ORBIT TYPES FOR PROPER LIE GROUPOIDS

JACK MORAVA

Abstract. It is widely understood that the quotient space of a topological group action has a complicated combinatorial structure, indexed somehow by the isotropy groups of the action [3 II §2.8]; but what, precisely, this structure might be seems unclear. This sketch defines a category of orbit types for a proper Lie groupoid (based on recent work [12-14] with roots in the theory of geometric quantization) as an attempt to capture some of this information.

1. Introduction and background

A topological groupoid or stack [10]

$$\mathsf{X} \;:=\; s,t \;: X_1 \rightrightarrows X_0$$

is **proper** if the map $s \times t : X_1 \to X_0 \times X_0$ is proper; such an object in the category of smooth manifolds and maps is a **proper Lie groupoid**. The quotient

$$\mathsf{X} \to \mathfrak{X}$$

of X_0 by the equivalence relation thus defined is a Hausdorff topological space, sometimes called the coarse moduli space of X.

Examples:

- Orbifolds [4]
- A topological **transformation group**, defined by a group action

$$G \times X \to X$$

has an associated topological groupoid $[X/G]$ with $X_0 = X,\ X_1 = G \times X$; I'll write X/G for its quotient space.

- **Toric varieties**, eg $G = \mathbb{T}^{n+1}/\mathbb{T} \cong \mathbb{T}^n$ acting on $X = \mathbb{C}P^n$ by

$$(u_0, \ldots, u_n) \cdot [z_0 : \cdots : z_n] = [u_0 z_0 : \cdots : u_n z_n] \;,$$

form a particularly nice class of examples. Their quotient objects are polytopes: in the case above $X/G \cong \Delta^n$ is a simplex. The faces of the polytope define a stratification [see §4.1 below] of the quotient, with the interiors of

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1991 *Mathematics Subject Classification.* **not ready for prime-time.**

the faces as strata. This defines an interesting poset, or category, associated to the groupoid: in this example it is the category of subsets of $\{0, \dots, n\}$ under inclusion.

An earlier paper [9] attempted to capture the sort of information encoded by the face poset of a toric variety, for more general group actions. The present note uses recent work on proper Lie groupoids to propose a more general construction[1].

Acknowledgement I am indebted to the organizers of the September 2013 Barcelona conference on homotopy type theory [1] for inspiration and hospitality, and for the opportunity to pursue these questions. I hope I will not be misunderstood by suggesting that classification problems of the sort considered here have a deep and nontrivial [15] history in philosophy.

2. Some technical preliminaries

2.1 Definition A reasonable space X has a universal map $X \to \pi_0 X$ to a discrete set, defined by the adjoint to the inclusion of the category of sets into that of topological spaces. The diagram

$$\begin{array}{ccccccc} \mathsf{X} : & & X_1 & \rightrightarrows & X_0 & \dashrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_0 \mathsf{X} : & & \pi_0 X_1 & \rightrightarrows & \pi_0 X_0 & \dashrightarrow & \pi_0 \mathfrak{X} \end{array}$$

extends π_0 to a functor from topological to discrete groupoids; for example

$$\pi_0[X/G] \cong [\pi_0(X)/\pi_0(G)] ,$$

with $\pi_0(X)/\pi_0(G) \cong \pi_0(X/G)$ for reasonable actions. There is moreover a natural transformation

$$\pi_0(\mathsf{X} \times \mathsf{Y}) \to \pi_0 \mathsf{X} \times \pi_0 \mathsf{Y} .$$

2.2 Regarding groups as categories with a single object defines a two-category (Gps) of groups. The set of homomorphisms from G_0 to G_1 has an action of G_1 by conjugation, defining a groupoid

$$\mathrm{Hom}_{\mathsf{Gps}}(G_0, G_1) := [\mathrm{Hom}(G_0, G_1)/G_1^{\mathrm{conj}}]$$

of morphisms from G_0 to G_1.

There are many variations on this theme, eg the topological two-category $(\mathsf{Gps})_{\mathrm{cpct}}$ defined by compact groups and continuous homomorphisms. It will be convenient to write (Gps^+) (resp. $(\mathsf{Gps}^+)_{\mathrm{cpct}}$) for the subcategories with such groups as objects, and spaces $\mathrm{Hom}_c^+(G_0, G_1)$ of continuous **one-to-one** homomorphisms as maps.

[1] See [7. 19] for approaches based on π_1 rather than π_0

The construction which is the identity on objects, and is the functor

$$[\mathrm{Hom}_c(G_0, G_1)/G_1^{\mathrm{conj}}] \to \mathrm{Hom}_{\pi_0 \mathsf{Gps}}(G_0, G_1) := \pi_0[\mathrm{Hom}_c(G_0, G_1)/G_1^{\mathrm{conj}}]$$

on morphism categories, defines a monoidal two-functor

$$(\mathsf{Gps})_{\mathrm{cpct}} \to \pi_0(\mathsf{Gps})_{\mathrm{cpct}}$$

[and similarly for $(\mathsf{Gps}^+)_{\mathrm{cpct}}$].

3. Groupoids of fixed-points with level structure

3.1 Definition: If X is a proper topological groupoid, and H is a compact Lie group, let

$$X(H)_0 := \{(x, \phi) \mid x \in X_0, \phi : H \to \mathrm{Iso}(x) \in \mathsf{Gps}^+_{\mathrm{cpct}}\}$$

and let $X(H)_1$ be the set of commutative diagrams of the form

$$\begin{array}{ccc} H & \overset{\phi'}{\rightarrowtail} & \mathrm{Iso}(x') \\ \vdots\, \gamma & & \downarrow g\text{−conj} \\ H & \overset{\phi}{\rightarrowtail} & \mathrm{Iso}(x) \end{array}$$

(with $g : x' \to x \in X_1$). The resulting proper topological groupoid $\mathsf{X}(H)$ is a model for the subgroupoid of X defined by points fixed by a group isomorphic to H. There is a forgetful morphism $\mathsf{X}(H) \to \mathsf{X}$, but it can't be expected to be the inclusion of a subgroupoid.

Proposition: $H \mapsto \mathsf{X}(H)$ defines a (two-)functor $\mathsf{X}(\bullet)$ from $\mathsf{Gps}^+_{\mathrm{cpct}}$ to the two-category $(\mathsf{Gpoids})_{\mathrm{cpct}}$ of proper topological groupoids.

Proof: First of all, if $\alpha : H_0 \rightarrowtail H_1 \in (\mathsf{Gps}^+_{\mathrm{cpct}})$ then

$$\begin{array}{ccccc} H_0 & \overset{\alpha}{\rightarrowtail} & H_1 & \overset{\phi_1'}{\rightarrowtail} & \mathrm{Iso}(x') \\ \vdots\, \gamma^\alpha & & & & \downarrow g\text{−conj} \\ H_0 & \rightarrowtail & H_1 & \overset{\phi_1}{\rightarrowtail} & \mathrm{Iso}(x) \end{array}$$

defines a functor

$$\alpha_{H_1}^{H_0} : \mathsf{X}(H_1) \to \mathsf{X}(H_0) .$$

Moreover, if $\alpha : H \to H$ is an inner automorphism of H (ie α is conjugation by $a \in H$) then there is a natural equivalence

$$\alpha_H^H \cong \mathbf{1}_{\mathsf{X}(H)}$$

defined by the commutative diagram

$$\begin{array}{ccccc} H & \xrightarrow{\alpha} & H & \xrightarrow{\phi} & \mathrm{Iso}(x) \\ \vdots & & & & \downarrow \phi(a^{-1}) \\ H & \xrightarrow{1_H} & H & \xrightarrow{\phi} & \mathrm{Iso}(x)\ . \end{array}$$

□

3.2 Claim: For any X as above, there is a commutative diagram

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & (\mathsf{Gpoids})_{\mathrm{cpct}*} \\ \downarrow \mathrm{Iso} & & \downarrow \\ (\mathsf{Gps})_{\mathrm{cpct}} & \xrightarrow{\mathsf{X}[\bullet]} & (\mathsf{Gpoids})_{\mathrm{cpct}} \end{array}$$

(with the category of pointed proper groupoids in the upper right corner, and the forgetful map to the category of proper groupoids along the right-hand edge). The left-hand vertical map sends $x \in X_0$ to its isotropy group, and the top horizontal map sends x to $\mathsf{X}(\mathrm{Iso}(x))$, with x as distinguished point.

Corollary The universal property of a fiber product defines a continuous functor from X to the category defined by the diagram

$$\begin{array}{ccccc} \Phi_0(\mathsf{X}) & \cdots\cdots> & (\mathsf{Gpoids})_{\mathrm{cpct}*} & \xrightarrow{\pi_0} & (\mathsf{Gpoids})_* \\ \vdots & & \downarrow & & \downarrow \\ (\mathsf{Gps})_{\mathrm{cpct}} & \xrightarrow{\mathsf{X}[\bullet]} & (\mathsf{Gpoids})_{\mathrm{cpct}} & \xrightarrow{\pi_0} & (\mathsf{Gpoids}) \end{array}$$

(with the two right vertical arrows being the obvious forgetful functors). □

It's natural to think of $\Phi_0(\mathsf{X})$ as a **database** category [9, 16]. However I do not know of a characterization of Φ_0 by some universal property.

3.3 Example: A proper transformation group $[X/G]$ defines a functor

$$X[*] : G > H \mapsto X^H = \{x \in X \mid \mathrm{Iso}(x) \subset H\}$$

on the topological category $(G-\mathrm{Orb})$ [with closed subgroups of G as objects, and

$\mathrm{Mor}_{G-\mathrm{Orb}}(H_0, H_1) = \mathrm{Maps}_G(G/H_0, G/H_1) = \{g \in G | gH_0g^{-1} \subset H_1\}/H_1^{\mathrm{conj}}$

as morphism objects [2 I §10].]

This extends to a functor

$$S^0[X^\bullet] : (G - \mathrm{Spaces}) \ni X \mapsto S^0[X^H] \in \mathrm{Func}(G - \mathrm{Orb}, S - \mathrm{Mod})$$

which provides a model [5 V §9, 18] for the G-equivariant stable category in terms of sheaves of spectra (ie S^0-modules) over $(G - \mathrm{Orb})$.

The commutative diagram

$$\begin{array}{ccc} [X/G] & \dashrightarrow & (\text{Sets})_* \\ \downarrow{\scriptstyle \text{Iso}} & & \downarrow \\ (G - \text{Orb}) & \xrightarrow{\pi_0 X[*]} & (\text{Sets}) \end{array}$$

defines a functor $\mathbf{\Phi}_0$ from $[X/G]$ to a fiber product category (with objects, pairs consisting of a subgroup H of G, and a component of X^H), analogous to the construction in the previous paragraph [9 §2.2]. The sheaf $S^\infty X^\bullet$ of spectra pulls back to a sheaf of spectra over $\mathbf{\Phi}_0(X)$.

One might hope for an unstable version of this construction, applicable in the theory of ∞-categories (cf eg [6 §5.5.6.18]); but because it depends on a presentation of $[X/G]$ as a global quotient, it does not seem to be homotopy-invariant.

4. Proper **Lie** groupoids, after Pflaum *et al*

4.1 A **stratification** $\mathcal{S}$ of a (paracompact, second countable) topological space X assigns to each $x \in X$, the germ of a closed subset $\mathcal{S}_x$ (containing x) of X. With suitably defined morphisms [8 §1.8, 14 §1], stratified spaces form a category. A stratification defines a locally finite partition

$$X = \coprod_{S \in \Sigma(\mathcal{S})} X_S$$

of X into locally closed subsets (called its strata), such that if $x \in X_S$ then $\mathcal{S}_x$ is the associated set germ.

Very interesting recent work of M. Pflaum *et al* [building on earlier work of Weinstein and Zung ([20]; cf also [11]) shows that

Theorem [14 Theorem 5.3, Cor 5.4] The quotient space $\mathfrak{X}$ of a proper Lie groupoid X has a canonical **Whitney** stratification. The associated decomposition of X_0 into locally closed submanifolds

$$X_{0(H)} \; = \; \{x \in X_0 \,|\, \text{Iso}(x) \cong H\}$$

is indexed [14 Theorem 5.7] by (isomorphism classes of) compact Lie groups H. □

4.2 Definition The **orbit** groupoid $\mathsf{O}(x) \subset \mathsf{X}$ of $x \in X_0$

$$\mathsf{O}_0(x) = \{y \in X_0 \mid \exists g : y \to x \;\in X_1\} n$$

$$\mathsf{O}_1(x) = \{g \in X_1 \mid s(g), t(g) \in \mathsf{O}_0(x)\}$$

reduces, in the case of a transformation groupoid $[X/G]$, to the groupoid

$$[(G/\text{Iso}(x))/G] \equiv [*/\text{Iso}(x)] \; .$$

A **slice** at $x \in X_0$ is (very roughly [14 §3.3-4,3.8-9]) the germ of an Iso(x)-invariant submanifold of X_0 containing x, transverse to $\mathsf{O}_0(x)$; for a transformation group it is something like the image of an exponential map

$$[\mathcal{N}_x/\mathrm{Iso}(x)] \equiv [(\mathcal{N}_x \times_{\mathrm{Iso}(x)} G)/G] \to [X/G]$$

(where $\mathcal{N}_x \in (\mathrm{Iso}(x) - \mathrm{Mod})$ is the linear representation

$$0 \to T_x G \to T_x X_0 \to \mathcal{N}_x \to 0 \ .$$

defining the normal bundle to the orbit of x).

Theorem [14 §3.11] There is an (essentially unique) slice at every object of a proper Lie groupoid X; the corresponding set germs define the canonical stratification [14 §5.4] of X.

Definition The **normal orbit type** of $x \in X_0$ is the equivalence class of its normal Iso(x)-representation $\mathcal{N}_x$. More precisely, $x_0 \sim x_1$ if there are isomorphisms

$$\phi : \mathrm{Iso}(x_0) \to \mathrm{Iso}(x_1), \ \Phi : \mathcal{N}_{x_0} \to \phi^*(\mathcal{N}_{x_1})$$

of groups and representations. The connected components $\nu \subset X_0$ of the normal orbit types of X are [14 §5.7] the strata of the canonical partition of X_0.

The **condition of the frontier** [14 Prop 5.15] asserts that if $\nu' \cap \overline{\nu} \neq \emptyset$ then $\overline{\nu} \supset \nu'$. This implies the existence of a partial order $(\nu > \nu')$ on the set $\Sigma(\mathsf{X})$ of connected components of normal orbit types for X, which can thus be regarded as the objects of a category [3 II §2.8].

4.3 Proposition For a proper Lie groupoid X, we have

$$\bigcup_{H<K} \ X_{0(K)} \times \mathrm{Hom}_c^+(H,K) \xrightarrow{\cong} X(H)_0$$

$$\bigcup_{H<K, x \in X_{0(K)}} \mathsf{O}_1(x) \times K \xrightarrow{\cong} X(H)_1$$

and consequently

$$\bigcup_{H<K} \ \mathfrak{X}_K \times \mathrm{Hom}_c^+(H,K)/K^{\mathrm{conj}} \xrightarrow{\cong} \mathfrak{X}(H)$$

(where $\mathfrak{X}_K \subset \mathfrak{X}$ is the space of orbits with isotropy group isomorphic to K).

4.4 Closing remarks

i) Since

$$\overline{X_{0(K)}} = \coprod_{X_{0(K)} \supset \nu > \nu'} \nu' \ ,$$

this gives **partial** control of $\pi_0\mathsf{X}(H)$ in terms of $\Sigma(\mathsf{X})$.

When $\mathsf{X} = [X/G]$ this all simplifies a little. In particular, since

$$X^H = \bigcup_{H<K<G} X_K \ ,$$

$\mathbf{\Phi}_0[X/G]$ is essentially just the quotient of $\Phi_0[X/G]$ which collapses the morphism spaces $\mathrm{Hom}_c^+(K,H)/K^{\mathrm{conj}}$.

ii) The subspaces $X_{0(K)}$ are disjoint unions of strata ν indexed by slice representations

$$K \to \mathrm{Aut}(\mathcal{N}_\nu) \ .$$

The resulting family of vector spaces over $\mathsf{X}(H)$ pulls back to a fibered category

$$\mathcal{N}(\mathsf{X}) \to \Phi_0(\mathsf{X}) \ .$$

This seems to provide a natural repository for Noether's theorem (which associates conserved quantities to elements of the Lie algebra of symmetries of states of a physical system) [9 §4.1].

iii) I don't know how generally one can associate a stratification to a topological groupoid. There are many interesting examples, coming from locally compact groupoids (eg the Thom-Boardman theory of singularities of smooth maps [17]), or from infinite-dimensional examples (Ebin's category of Riemannian metrics up to diffeomorphism, Vassiliev's finite-type invariants of immersions, ...), where a good general theory might be useful. The existence of slices would seem to be an essential requirement for such a theory.

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