The Adjoint Action of a Homotopy-associative H-space on its Loop Space.

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Assumptions

Unless specified,

- All topological spaces:
  - have the homotopy type of a CW complex of finite type
  - possess a basepoint.
- All maps between spaces are continuous and respect the basepoint.

Assumptions

- Algebras have $\mathbb{F}_p$ (p a fixed odd prime) as their base field.
- All algebras and rings will be associative, graded, and finitely generated in each degree.
- A homomorphism between algebras means a graded algebra homomorphism.

Outline

- Adjoint Action of a Lie Group on its Loop Space
- Homotopy-associative H-spaces
- Defining the Adjoint Action for Homotopy-associative H-spaces
- Characterizing Homology

Question in Homology

The Lie group multiplication map $\mu : G \times G \to G$ induces an algebra structure in homology:

$$\mu_* : H_*(G; \mathbb{F}_p) \otimes H_*(G; \mathbb{F}_p) \to H_*(G; \mathbb{F}_p)$$

The homology $H_*(G; \mathbb{F}_p)$ is an associative algebra with multiplication μ.

Question in Homology

- If G is abelian (up to homotopy), $H_*(G;\mathbb{F}_p)$ is (graded) commutative.
- Is the converse true?

Examples

- If G is any torus, then G abelian and $H_*(G,\mu;\mathbb{F}_3)$ is commutative
- If $G = F_4$, then G is not abelian. Also, $H_*(G,\mu;\mathbb{F}_3)$ is not commutative
- If $G = Sp(4)$, then G is abelian, but $H_*(G,\mu;\mathbb{F}_3)$ is commutative

Adjoint Action of a Lie Group on Its Loop Space

Definition

The adjoint action $Ad : G \times \Omega G \to \Omega G$ of a Lie group G on its loop space is given by

$$Ad(g, l)(t) = gl(t)g^{-1}$$

where $g \in G$, $l \in \Omega G$, and $0 \leq t \leq 1$.

Applications

Theorem (Kono, Kozima)

Let G be a compact simply-connected Lie group and $p_2 : G \times \Omega G \to \Omega G$ be projection onto the second factor. Then the induced homomorphisms

$$Ad^* : H^*(\Omega G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p) \otimes H^*(\Omega G; \mathbb{F}_p)$$

$$p_2^* : H^*(\Omega G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p) \otimes H^*(\Omega G; \mathbb{F}_p)$$

are equal if and only if the algebra $H_*(G; \mathbb{F}_p)$ is commutative.

Defining the Adjoint Action Beyond Lie Groups

This talk will answer:

- how to define *Ad* for a generalization of Lie groups which may not have strict associativity or inverses
- how this definition of *Ad* can be used to prove the previous theorem for these spaces

H-spaces

Definition

Let X be a topological space with basepoint x_0. Suppose we have a map $\mu : X \times X \to X$ such that

$$\mu(x, x_0) = \mu(x_0, x) = x$$

$$xx_0 = x_0x = x$$

Then X is called an *H-space*. The basepoint x_0 is called a *strict identity*.

H-spaces

- No assumptions of associativity or inverses
- Adams: There exist infinitely many H-spaces which are not topological groups

Homology of H-spaces

- For any H-space X, its homology $H_*(X;\mathbb{F}_p)$ is a ring with multiplication given by

$$\mu_* : H_*(X;\mathbb{F}_p) \otimes H_*(X;\mathbb{F}_p) \to H_*(X;\mathbb{F}_p),$$

$$\mu_*(\bar{x} \otimes \bar{y}) = \bar{x}\bar{y}.$$

- The ring might not be commutative or associative.

Homotopy-Associative H-spaces

Definition

An H-space X is said to be *homotopy-associative* if these are homotopic:

$$(x, y, z) \mapsto x(yz)$$

$$(x, y, z) \mapsto (xy)z$$

Homotopy Inverse Operations

Definition

Given an H-space X a *(two sided) homotopy inverse operation* is a map $i : X \to X$ such that these are homotopic:

$$x \mapsto xi(x)$$

$$x \mapsto i(x)x$$

$$x \mapsto x_0$$

HA-spaces

Definition

An H-space which is homotopy associative and has a two-sided homotopy inverse operation will be called an *HA-space*.

Examples of HA-spaces

- Lie Groups and Topological Groups
- Adams: There exist infinitely many HA-spaces which are not topological groups

Homology of HA-spaces

- For any HA-space X, its homology $H_*(X;\mathbb{F}_p)$ is a ring with multiplication given by

$$\mu_* : H_*(X;\mathbb{F}_p) \otimes H_*(X;\mathbb{F}_p) \to H_*(X;\mathbb{F}_p),$$

$$\mu_*(\bar{x} \otimes \bar{y}) = \bar{x}\bar{y}.$$

- Homotopy-associativity of X implies $H_*(X;\mathbb{F}_p)$ is an associative ring.

Adjoint Action Definition for an HA-space?

Definition? Let X be a finite simply-connected HA-space. Can we define $Ad : X \times \Omega X \to \Omega X$ pointwise by

$$Ad(x,l)(t) = (xl(t))\, i\,(x(t))?$$

Problem: At $t = 0$, $l(0) = x_0$, so

$$Ad(x,l)(0) = xi(x).$$

X may not have a strict inverse, so $Ad(x,l)$ might not be in ΩX.

Free Loop Space

Definition

Given a simply-connected space X, the free loop space of X, ΛX, is

$$\Lambda X = \{\alpha : [0,1] \to X : \alpha(0) = \alpha(1)\}$$

Modified Codomain

Definition

We define $\widehat{Ad} : X \times \Omega X \to \Lambda X$ so that given $x \in X$ and $l \in \Omega X$, for any $t \in [0,1]$,

$$\widehat{Ad}(x,l)(t) = (xl(t))\, i\,(x(t)) \, .$$

An Important Fibration

Definition

Let X be any simply-connected HA-space, $j : \Omega X \to \Lambda X$ be the inclusion

$$j(l) = l,$$

and $\varepsilon_0 : \Lambda X \to X$ be evaluation at $t = 0$:

$$\varepsilon_0(\varphi) = \varphi(0).$$

An Important Fibration

$$\varepsilon_0(\varphi) = \varphi(0)$$

Definition

The map ε_0 is a fibration, and we have a fibration sequence which we call the *free loop fibration*:

$$\begin{array}{c} \Omega X \\ \Big\downarrow j \\ \Lambda X \\ \Big\downarrow \varepsilon_0 \\ X \end{array}$$

The Inclusion Map j and Lifts

Lemma (Nguyen)

Let Y be any path-connected topological space, X be any simply-connected HA-space, and suppose we are given maps $f_1, f_2 : Y \to \Omega X$, such that

$$jf_1 \simeq jf_2.$$

Then

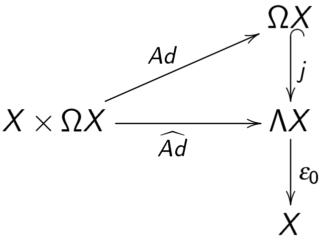
$$f_1 \simeq f_2.$$

Note: The map j is not a fibration, and the second homotopy is not a lift of the first.

Defining *Ad* for HA-spaces

Definition (Nguyen)

There is a map $Ad : X \times \Omega X \to \Omega X$ (unique up to homotopy) such that the diagram commutes up to homotopy:



A Property of *Ad*

- Kono, Kozima: Given a finite simply-connected Lie group G,

$$Ad\,(id_G \times Ad) = Ad\,(\mu \times id_{\Omega G})$$

A Property of *Ad*

- Kono, Kozima: Given a finite simply-connected Lie group G,

$$Ad\,(id_G \times Ad) = Ad\,(\mu \times id_{\Omega G})$$

- My definition satisfies

$$Ad\,(id_X \times Ad) \simeq Ad\,(\mu \times id_{\Omega X})$$

for any simply-connected HA-space X.

Main Result

Theorem (Nguyen)

Let X be a finite simply-connected HA-space and $p_2 : X \times \Omega X \to \Omega X$ be projection onto the second factor. Then the induced homomorphisms

$$Ad^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$$

$$p_2^* : H^*(\Omega X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes H^*(\Omega X; \mathbb{F}_p)$$

are equal if and only if the algebra $H_*(X; \mathbb{F}_p)$ is commutative.

Example (non-finite space)

- Adams: There is a simply-connected HA-space X for which
$$H_*(X;\mathbb{F}_5) \cong \wedge(\bar{x}_5).$$
We can use the property to show that
$$Ad_* = p_{2*}.$$

Open Questions

- If $H_*(X; \mathbb{F}_p)$ is not commutative, what does Ad^* or Ad_* look like?
- Can we remove the finiteness assumption?

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