Power Operations and Commutative Ring Spectra

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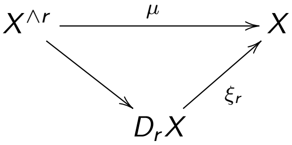
Goals

- Compute power operations
- Interpret computations
- Motivate study of relative smash products

1. Introduction
2. Künneth Spectral Sequence
3. Computations
   - Complex Connective K-theory
   - $BP\langle 2\rangle$ at the prime 2
   - Complex Cobordism
4. Interpretations

Commutative $\mathbb{S}$-algebras and Power Operations

- We have well behaved categories of commutative $\mathbb{S}$-algebras and module spectra over them,([EKMM]).
- The relative smash product is the pushout in commutative $\mathbb{S}$-algebras.
- The product of a commutative $\mathbb{S}$-algebra factors through the extended power (or homotopy orbit) construction.



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$$\begin{array}{ccc} X^{\wedge r} & \xrightarrow{\mu} & X \\ & \searrow \quad \nearrow_{\xi_r} & \\ & D_r X & \end{array}$$

(These give power operations)

More Facts

- Power operations are preserved by commutative $\mathbb{S}$-algebras maps.
- A commutative E-algebra has E theoretic operations.
- McClure, Mandell $\rightsquigarrow E = \mathrm{H}\mathbb{F}_p$. $(Q^i : H_n(X) \to H_{i+n}(X))$
- Bruner $\rightsquigarrow E = S^0$.
- McClure $\rightsquigarrow E = K_p^\wedge$.
- Rezk ($p = 2$), Zhu ($p = 3$) $\rightsquigarrow E = E_2$.
- tom Dieck $\rightsquigarrow E = MU$.

Computing Relative Smash Products

Theorem (T.,EKMM)

Let R a commutative $\mathbb{S}$-algebra, A, B be right and left R-modules respectively.

- $E_2^{p,q} = \mathrm{Tor}_q^{\pi_* R}(\pi_* A, \pi_* B)_p \Longrightarrow \pi_{p+q}(A \wedge_R B)$.
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Use a comparison theorem for filtrations. □

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Proof.

Use a comparison theorem for filtrations. □

The Künneth spectral sequence also supports a theory of power operations.

The above allows us to use the KSS to compute the action of the DL algebra on $\pi_*(\mathrm{H}\mathbb{F}_2 \wedge_R \mathrm{H}\mathbb{F}_2)$.

- Compute $\mathrm{Tor}_*^{R_*}(\mathbb{F}_2, \mathbb{F}_2)_* \Rightarrow \pi_* \mathrm{H}\mathbb{F}_2 \wedge_R \mathrm{H}\mathbb{F}_2$.
- Compute $\mathrm{Tor}_*^{R_*}(\mathrm{H}\mathbb{F}_{2*}R, \mathbb{F}_2)_* \Rightarrow \mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$.
- Compare the two to compute operations in $\mathrm{H}\mathbb{F}_2 \wedge_R \mathrm{H}\mathbb{F}_2$.

$$\begin{array}{ccc} \mathrm{Tor}_s^{R_*}(\mathrm{H}\mathbb{F}_{2*}R, \mathrm{H}\mathbb{F}_{2*})_t & \Longrightarrow & \mathrm{H}\mathbb{F}_{2_{s+t}}\mathrm{H}\mathbb{F}_2 \\ \downarrow \check{\phi} & & \downarrow \check{\phi} \\ \mathrm{Tor}_s^{R_*}(\mathbb{F}_2, \mathbb{F}_2)_t & \Longrightarrow & \pi_{s+t}(\mathrm{H}\mathbb{F}_2 \wedge_R \mathrm{H}\mathbb{F}_2). \end{array}$$

Recollections

Theorem (Milnor)

The dual Steenrod algebra is $\mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2 \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$.

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Remark

- $\mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$ *is a Hopf algebra,* $\chi(\xi_i) = \overline{\xi}_i$
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Theorem (Steinberger)

- $Q^{2^i-2}(\xi_1) = \overline{\xi}_i \in \mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$
- $Q^{2^i}(\overline{\xi}_i) = \overline{\xi}_{i+1} \in \mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$ *for* $i \geq 1$

Facts about ku

- ku is a commutative $\mathbb{S}$-algebra.
- $\pi_* ku \cong \mathbb{Z}[v]$ with $|v| = 2$.
- $\mathrm{H}\mathbb{F}_{2*} ku$ is a trivial ku_*-module.
- $\mathrm{H}\mathbb{F}_{2*} ku \cong \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3, \bar{\xi}_4, \ldots]$.

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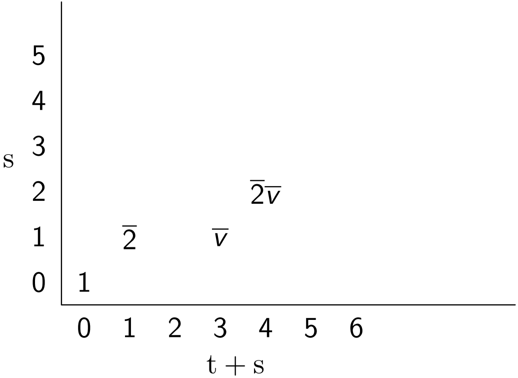
$$\begin{array}{ccc} \mathrm{Tor}_s^{ku_*}(H\mathbb{F}_{2*}ku, \mathbb{F}_2)_t & \Longrightarrow & H\mathbb{F}_{2\,s+t}H\mathbb{F}_2 \\ \Big\downarrow \check{\phi} & & \Big\downarrow \check{\phi} \\ \mathrm{Tor}_s^{ku_*}(\mathbb{F}_2, \mathbb{F}_2)_t & \Longrightarrow & \pi_{s+t}(H\mathbb{F}_2 \wedge_{ku} H\mathbb{F}_2). \end{array}$$

$\mathrm{Tor}_*^{ku_*}(\mathbb{F}_2, \mathbb{F}_2)_* \Rightarrow \pi_* H\mathbb{F}_2 \wedge_{ku} H\mathbb{F}_2$



$\mathrm{Tor}_*^{ku_*}(H\mathbb{F}_{2*}ku, \mathbb{F}_2)_* \Rightarrow H\mathbb{F}_{2*}H\mathbb{F}_2$

s \ t + s	0	1	2	3	4	5	6	7
5								
4								
3								
2					$\overline{2}\overline{v}$			
1		$\overline{2}$		$\overline{v}, \overline{2}\xi_1^2$		$\cdots$		
0	1		ξ_1^2		ξ_1^4		ξ_2^2	ξ_3

We compute the action of the Dyer-Lashof algebra as follows.

- $Q^2(\xi_1) = \overline{\xi}_2$in $\mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$.
- $\overline{\xi}_2 = \xi_1^3 + \xi_2$ in $\mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$.
- $\overline{2}$ detects ξ_1 in the spectral sequence converging to $\mathrm{H}\mathbb{F}_{2*}\mathrm{H}\mathbb{F}_2$.
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Proposition (T.)

$\mathrm{Tor}_*^{ku_*}(\mathbb{F}_2, \mathbb{F}_2)_* \Rightarrow \pi_*\mathrm{H}\mathbb{F}_2 \wedge_{ku} \mathrm{H}\mathbb{F}_2$ *collapses at* E_2.
$\pi_*\mathrm{H}\mathbb{F}_2 \wedge_{ku} \mathrm{H}\mathbb{F}_2 \cong E[\overline{2}, \overline{v}]$ *with* $Q^2(\overline{2}) = \overline{v}$ *where* $|\overline{2}| = 1$ *and* $|\overline{v}| = 3$.

Facts about Lawson and Naumann's $BP\langle 2\rangle$

- $BP\langle 2\rangle$ is a commutative $\mathbb{S}$-algebra at the prime 2.
- $\pi_* BP\langle 2\rangle \cong \mathbb{Z}_{(2)}[v_1, v_2]$ with $|v_1| = 2$ and $|v_2| = 6$.
- $\mathrm{H}\mathbb{F}_{2*} BP\langle 2\rangle$ is a trivial $BP\langle 2\rangle_*$-module.
- $\mathrm{H}\mathbb{F}_{2*} BP\langle 2\rangle \cong \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \bar{\xi}_5, \ldots]$.

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Proposition (T.)

- $\pi_* \mathrm{H}\mathbb{F}_2 \wedge_{BP\langle 2\rangle} \mathrm{H}\mathbb{F}_2 \cong E[\overline{2}, \overline{v_1}, \overline{v_2}]$.
- $Q^2(\overline{2}) = \overline{v_1}$, $Q^6(\overline{2}) = \overline{v_2}$, $Q^4(\overline{v_1}) = \overline{v_2}$, *and* $Q^6(\overline{2}\overline{v_1}) = \overline{v_1 v_2}$ *where* $|\overline{2}| = 1$, $|\overline{v_1}| = 3$ *and* $|\overline{v_2}| = 7$.

Facts about MU

- MU is a commutative $\mathbb{S}$-algebra.
- $\pi_* MU \cong \mathbb{Z}[x_1, x_2, \ldots]$ with $|x_i| = 2i$.
- $\mathrm{H}\mathbb{F}_{2*} MU$ is not a trivial MU_*-module, but it is manageable.
- $\mathrm{H}\mathbb{F}_{2*} MU \cong P \otimes \mathrm{H}\mathbb{F}_{2*} BP$.

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- $Q^{2^i - 2}(\overline{2}) = \overline{x}_{2^{i-1}-1}$, $Q^{2^i}(\overline{x}_{2^{i-1}-1}) = \overline{x}_{2^i - 1}$ *where* $|\overline{2}| = 1$ *and* $|\overline{x}_n| = 2n + 1$.

This computation has the following corollary.

Corollary (T.)

Let I be an ideal of MU_* generated by a regular sequence. If I contains a non-zero finite number of the x_{2^i-1}, then the quotient map $MU \to MU/I$ cannot be realized as a map of commutative $\mathbb{S}$-algebras.

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Proof.

Suppose there were such a $MU \to MU/I$. This induces

$$\mathrm{H}\mathbb{F}_2 \wedge_{MU} \mathrm{H}\mathbb{F}_2 \longrightarrow \mathrm{H}\mathbb{F}_2 \wedge_{MU/I} \mathrm{H}\mathbb{F}_2$$

which must preserve power operations, but it can't. □

What does $Q^2(\overline{2}) = \overline{v}$ tell us? $\overline{2}$ is the "difference" of two null-homotopies of 2.

$$S^0 \xrightarrow{2} ku$$

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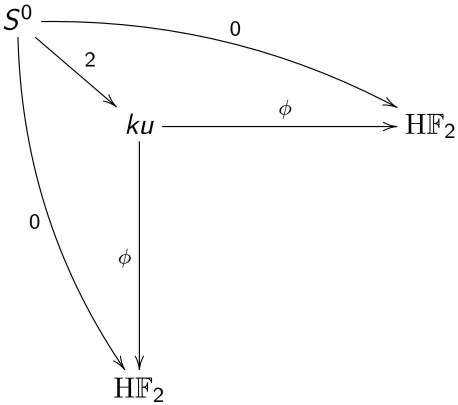

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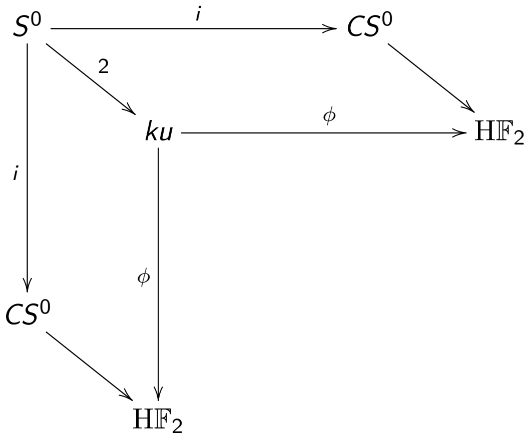

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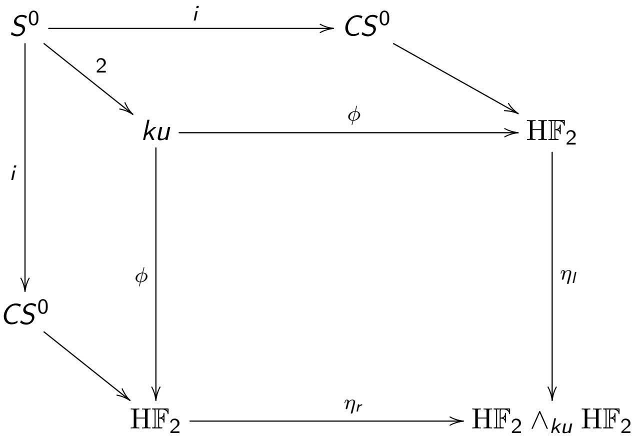

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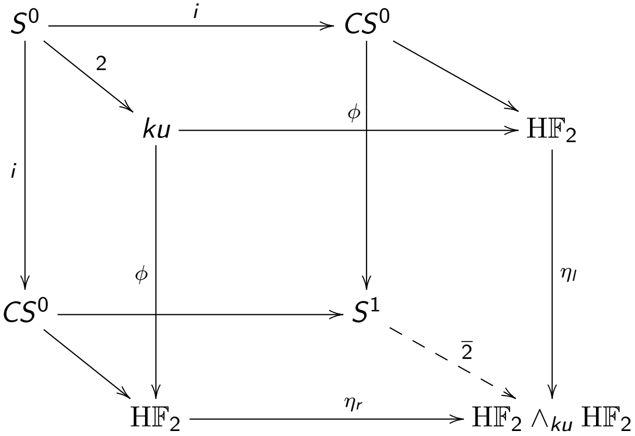

Given maps of commutative $\mathbb{S}$-algebras

$$\begin{array}{ccc} ku & \xrightarrow{\ \phi\ } & H\mathbb{F}_2 \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle f} \\ H\mathbb{F}_2 & \xrightarrow{\ g\ } & X. \end{array}$$

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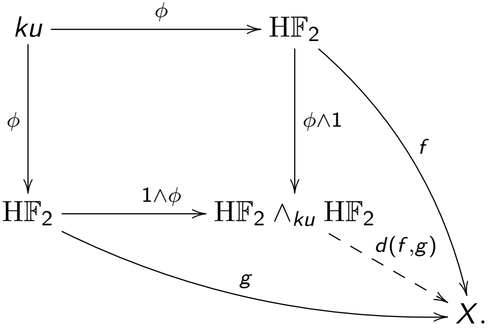

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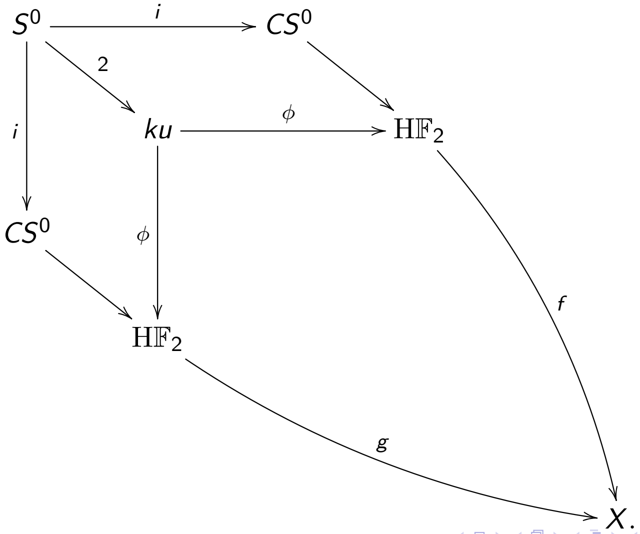

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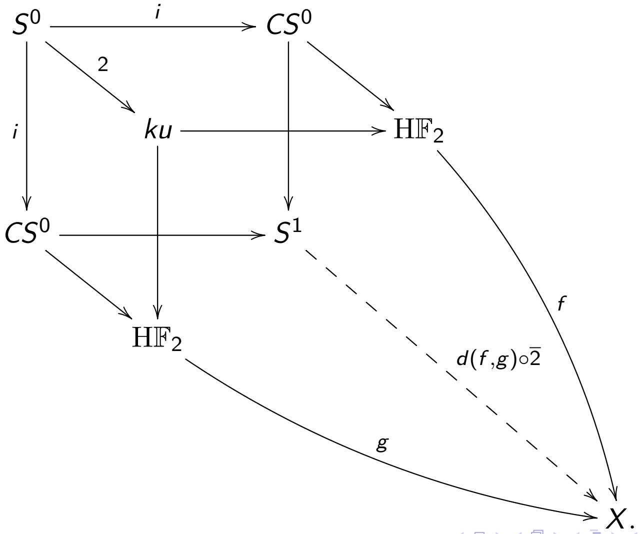

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- If f & g cone off 2 in the same way then they must cone off v in the same way.
- While there is no homotopy operation relating 2 & v in $\pi_* ku$, they are related.
- By work of Mandell, $\mathrm{H}\mathbb{F}_2 \wedge_{ku} \mathrm{H}\mathbb{F}_2$ can be thought of as an E_∞-dga. By the above, it detects some of the E_∞-structure of ku.

In general, we have the following result.

Theorem (T.)

If $\phi : R \to A$ is a map of commutative $\mathbb{S}$-algebras that is surjective in homotopy. Then, $\forall x \in I := ker(\phi_*)$ with nonzero image in I/I^2 there is a nonzero class $\overline{x} \in \mathrm{Tor}_1^{R_*}(A_*, A_*)$. If this class is not an "eventual" boundary in the Künneth spectral sequence, then it can be realized as the difference of two null-homotopies of $\phi_*(x) \in A_*$.

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Proposition (T.)

If $\phi : R \to A$ is a map of commutative $\mathbb{S}$-algebras over $\mathrm{H}\mathbb{F}_2$. If $x \in ker(\phi_*)$ and $Q^i(\overline{x}) = \overline{y} \in \mathrm{Tor}_1^{R_*}(\mathbb{F}_2, \mathbb{F}_2)$, then $\phi_*(y) \in \pi_* A$ is decomposable.