Enriched Morita Theory

An Enriched Perspective on Morita Theory
With a View Toward Bicategorical Contexts

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Goals

As we talk about Morita theory ...

- Introduce the language and ideas of enriched category theory
- Explain standard equivalences as enriched equivalences
- Advertise the bicategorical Yoneda lemma

Outline

Morita Theory for Rings

Theorem (Morita)

Let R and S be rings. Then $\mathit{Mod}\text{-}R \simeq \mathit{Mod}\text{-}S$ if and only if there exists an R-S-bimodule, ${}_RP_S$, such that:

- P_S is a finitely-generated and projective S-module.

  $\Rightarrow P \otimes_S \mathrm{Hom}_S(P, S) \xrightarrow{\cong} \mathrm{Hom}_S(P, P)$
- P_S generates $\mathit{Mod}\text{-}S$.

  'generator' means every S-module can be resolved by P
  e.g. S generates $\mathit{Mod}\text{-}S$
  $\Rightarrow \mathrm{Hom}_S(P, S) \otimes_R P \xrightarrow{\cong} S$
- $\mathrm{Hom}_S({}_RP_S, {}_RP_S) \cong {}_RR_R$ as an R-R-bimodule.

  We could take P to be a right S-module and define $R = \mathrm{Hom}_S(P, P)$.

Morita Theory for Rings

Idea of proof

Suppose $F : \mathit{Mod}\text{-}R \xrightarrow{\simeq} \mathit{Mod}\text{-}S$ and let $P = F(R_R)$. Then

$$ {}_R R_R \cong \mathrm{Hom}_R(R, R) \xrightarrow{\cong} \mathrm{Hom}_S(F(R), F(R)) = \mathrm{Hom}_S(P, P). $$

Thus, P has the structure of an R-S-bimodule.

R_R finitely generated and projective $\Rightarrow$
P_S finitely generated and projective.

$$ P \otimes_S \mathrm{Hom}_S(P, S) \xrightarrow{\cong} \mathrm{Hom}_S(P, P) $$

R_R generates $\mathit{Mod}\text{-}R \Rightarrow$
P_S generates $\mathit{Mod}\text{-}S$.

$$ \mathrm{Hom}_S(P, S) \otimes_R P \xrightarrow{\cong} S $$

The functor $- \otimes_R P$ is an equivalence of categories; its adjoint is $\mathrm{Hom}_S(P, -)$.

Such a functor is called a **standard functor**.

Standard Functors Are Enriched Functors

Two important things to note:

- The isomorphism $\mathrm{Hom}_R(R, R) \cong \mathrm{Hom}_S(F(R), F(R))$ is an isomorphism of Abelian groups.

Fact

Every functor which has a right adjoint preserves Abelian group structure on Hom sets.

- The standard functor, $- \otimes_R P$, preserves all left-module structure. For modules X and Y, over rings C and D,

$$({}_D X_C \otimes_C {}_C Y_R) \otimes_R P \cong {}_D X_C \otimes_C ({}_C Y_R \otimes_R P)$$

Categories

A category, $\mathscr{C}$, has ...

1. Objects
2. Morphisms
   - Composition pairing: $\mathscr{C}(Y,Z) \times \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$
   - Composition is associative
   - Composition is unital (identity morphisms)

A category, $\mathcal{V}$, is **monoidal** if it has a tensor product:

- $X, Y \in \mathcal{V} \Rightarrow X \otimes Y \in \mathcal{V}$
- $X_1 \xrightarrow{f_1} Y_1$ and $X_2 \xrightarrow{f_2} Y_2 \Rightarrow X_1 \otimes X_2 \xrightarrow{f_1 \otimes f_2} Y_1 \otimes Y_2$
- associative and unital
- compatible with composition of morphisms

Enriched Categories

Enriched categories are categories in which the collection of morphisms between two objects is not just a set.

A category, $\mathscr{C}$, is **enriched** in a monoidal category, $\mathcal{V}$, if between every two objects $X, Y \in \mathscr{C}$ there is a hom object $X \triangleright Y \in \mathcal{V}$.

These hom objects need to have a composition pairing

- Uses the monoidal structure of $\mathcal{V}$:
  a morphism in $\mathcal{V}$ $(Y \triangleright Z) \otimes (X \triangleright Y) \to X \triangleright Z$.
- Associativity and unitality of composition: diagrams relating to the associtivity and unitality of $\otimes$.

A category, $\mathcal{W}$, is a **closed monoidal category** if it is a monoidal category enriched in itself, with an adjunction between $\otimes$ and $\triangleright$:

$$\mathcal{W}(X \otimes Y, Z) \cong \mathcal{W}(X, Y \triangleright Z)$$

naturally for all $X, Y, Z \in \mathcal{W}$.

Outline

Modules Over a Commutative Ring, k

- *Mod*-k is a monoidal category.
  The commutativity of k allows us to define a k-module structure on $M \otimes_k N$ for right k-modules M and N.
- *Mod*-k is a closed monoidal category.
  $\mathrm{Hom}_k(M, N)$, likewise, has a k-module structure.

Modules over Noncommutative Rings

If R is a non-commutative ring, then *Mod*-R is not a monoidal category, nor is it closed.

However, *Mod*-R is enriched in Abelian groups.

Moreover, if ${}_C X_R$ and ${}_D Y_R$ are right R-modules, and left-modules over rings C and D, then $X \triangleright Y = \mathrm{Hom}_R(X, Y)$ is a C-D-bimodule.

Bimodules

Let $Mod(R,S)$ denote $Mod\text{-}R \otimes S^{op}$; the category of S-R-bimodules.

Tensor product of modules gives a pairing

$$\begin{array}{ccc} Mod(R,S) \times Mod(S',R) & \longrightarrow & Mod(S',S) \\ ({}_S M_R, {}_R L_{S'}) & \mapsto & M \otimes_R L \end{array}$$

Associativity and unitality for $\otimes$ makes *Mod* a category enriched in $\mathcal{C}at$!

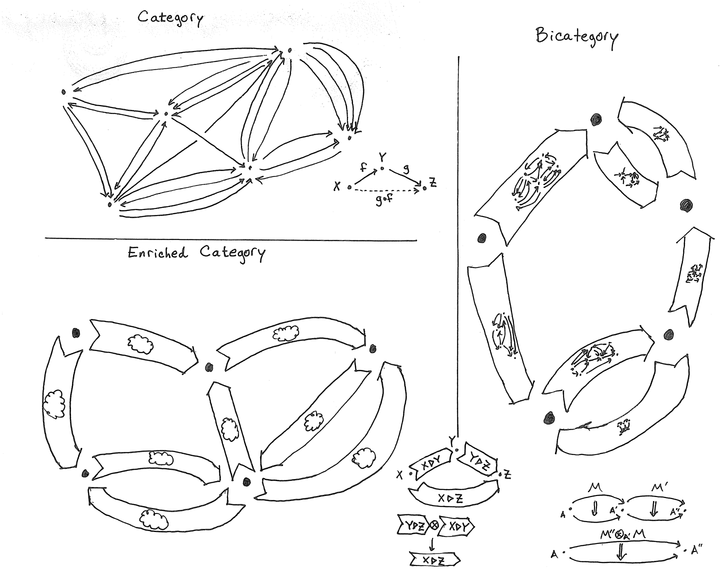
A **bicategory** is a category enriched in the monoidal category $\mathcal{C}at$.

Caveat: For exposition, we neglected the difference between *strict* enrichments and *weak* enrichments. A bicategory is weakly enriched in $\mathcal{C}at$ because the associativity and unit conditions for composition (tensor product) only hold up to isomorphism, not equality. This difference does not affect our discussion, but for those trying to understand the distinction between bicategories and 2-categories, it is an important point.

Picture

This is an attempt to depict some of the important features in the description of enriched categories and bicategories. Note, in the picture of a category, that there may be many morphisms between two given objects. In the picture of an enriched category, there is a single hom *object* from one object to another. These are labeled with small clouds to indicate that they might be objects of any monoidal category. The small diagram near the bottom shows how composition requires a monoidal structure on the enriching category.

A bicategory is a category enriched in $\mathcal{C}at$, and so here we have drawn categories on each of our hom objects. An alternate depiction shows the objects of a hom category as 1-cells, and the morphisms of a hom category as 2-cells. We use this to depict the composition, which is given by tensor product.



Outline

The Bicategory *Mod*

A bicategory is a category enriched `(weakly)` in *Cat*

- 0-cells Rings
- 1-cells Bimodules
- 2-cells Maps of bimodules

For every pair of rings, A and B, $Mod(A, B)$ is an enriched category.

The bicategory *Mod* has a closed structure

For $M \in Mod(R, S)$ and $N \in Mod(R, S')$, $M \triangleright N \in Mod(S, S')$.

$M \triangleright N = \mathrm{Hom}_R(M, N)$

The composition pairing $(Y \triangleright Z) \otimes (X \triangleright Y) \to X \triangleright Z$ is defined by composition of bimodule maps;
X, Y, and Z are right R-modules.

Standard Transformations in *Mod*

- Recall: The standard functor, $- \otimes_R P$, preserves all left-module structure.

  This means that, for any ring C, $- \otimes_R P$ defines a functor

  $$Mod(R, C) \to Mod(S, C).$$

- $Mod(R, -)$ and $Mod(S, -)$ are **represented pseudofunctors**
  - Take, as input, other 0-cells of *Mod*. (rings)
  - Give, as output, categories. (categories of bimodules)
  - For a 1-cell $K : C \to C'$ (bimodule),
    have functor $Mod(R, C) \to Mod(R, C')$ given by $K \otimes_C -$.
- The 'standard functors' are called **strong transformations** of represented pseudofunctors.

  families of functors (one for each ring, C)

Natural Transformations of Represented Functors

Let $\mathscr{C}$ denote a category.

For an object A, $\mathscr{C}(A,-)$ is a represented functor $\mathscr{C} \to \mathcal{S}et$.

A natural transformation of represented functors $\mathscr{C}(A,-) \to \mathscr{C}(B,-)$ is a family of maps in $\mathcal{S}et$ (**components**):

$$\eta_C : \mathscr{C}(A,C) \to \mathscr{C}(B,C)$$

for each object $C \in \mathscr{C}$.

- Subject to compatibility with morphisms $C \to C'$ See Diagram

Yoneda Lemma

The set of natural transformations $\mathscr{C}(A,-) \to \mathscr{C}(B,-)$ is isomorphic to the set of morphisms $\mathscr{C}(B,A)$.

Every natural transformation of represented functors is given by pull-back along a morphism $f : B \to A$.

Now consider the bicategory Mod.
A strong transformation of represented pseudofunctors $Mod(R, -) \to Mod(S, -)$ is a family of functors (**components**):

$$H_C : Mod(R, C) \to Mod(S, C)$$

for each ring C.

- Subject to compatibility with 1-cells $C \to C'$. See Diagram

Bicategorical Yoneda Lemma

The category of strong transformations $Mod(R, -) \to Mod(S, -)$ is equivalent to the category of 1-cells $Mod(S, R)$.

Every compatible family of functors is given by a 'standard functor'.

Morita Theory is Implicitly a Theory of Enriched Equivalences.

- The 'standard functors' $(- \otimes_R P)$ preserve closed structure.
- Recall: Every equivalence between categories of modules preserves enrichment.

  does so compatibly with composition and units

Fact for General Enrichments

Every family of functors which jointly preserves closed structure is a strong transformation of represented pseudofunctors.

Corollary (Morita II)

Every equivalence of module categories is a standard equivalence.

For more general enrichments, not all equivalences preserve enrichments.

Morita Theory in Other Closed Bicategories

Questions Morita theory studies:

- Which standard functors are equivalences?

  for *Mod*: finitely-generated projective generators

- When are categories equivalent via a standard equivalence?

  standard equivalences yield invariants

An enriched/bicategorical perspective observes:

- Standard functors preserve ambient enrichment and ambient bicategorical structure.

  bicategorical Yoneda lemma

- Bicategorical framework provides a context in which to study standard equivalences.

These diagrams, drawn on the side board during the talk, accompany the descriptions of transformations.

Natural transformation of represented functors on a category $\mathscr{C}$. Return

$$\begin{tikzcd} C \arrow[d, "K"'] \\ C' \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}(A, C) \arrow[r, "\eta_C"] \arrow[d, "K_*"'] & \mathscr{C}(B, C) \arrow[d, "K_*"'] \\ \mathscr{C}(A, C') \arrow[r, "\eta_{C'}"] & \mathscr{C}(B, C') \end{tikzcd}$$

$$\eta_{C'}(K_*(\varphi)) = K_*(\eta_C(\varphi))$$

or, more simply,

$$\eta(K \circ \varphi) = K \circ \eta(\varphi)$$

for $\varphi \in \mathscr{C}(A, C)$.

Strong transformation of represented pseudofunctors on the bicategory *Mod*. Return

$$\begin{tikzcd} C \arrow[d, "K"'] \\ C' \end{tikzcd} \qquad \begin{tikzcd} Mod(R, C) \arrow[r, "H_C"] \arrow[d, "K_*"'] & Mod(S, C) \arrow[d, "K_*"'] \\ Mod(R, C') \arrow[r, "H_{C'}"] & Mod(S, C') \end{tikzcd}$$

$$H(K \otimes_C X) \cong K \otimes_C H(X)$$

for $X \in Mod(R, C)$.