Morita theory and Azumaya objects in bicategorical contexts

Niles Johnson

University of Chicago

http://www.math.uchicago.edu/~niles

January 2009
Joint Meetings, Washington, D.C.

Abstract

- Morita theory: bicategorical structure in classic algebra
- Higher-categorical Folklore: generalizations of the Picard group, Azumaya algebras, Brauer group
- Algebraic theory provides 'hands-on' descriptions

Goals

Develop calculationally accessible description of Morita theory.

- Characterization of Standard Morita Equivalences
  Induced by invertible 1-cells in a bicategory
- Characterization of Azumaya Objects
  Brauer groups

Tools: duality and Yoneda

A Bicategory is a Weak 2-Category

A bicategorical context provides:

- organizational framework
- conceptual advantage

Main example: $\mathscr{M}_R$ is the bicategory of algebras and bimodules over a commutative ring, R

- 0-cells: R-algebras
- 1-cells: bimodules $\mathscr{M}_R(A, B)$ is the category of (B, A)-bimodules
- 2-cells: bimodule morphisms

The horizontal composite of 1-cells is given by the tensor product

$$N \otimes_B M : A \xrightarrow{M} B \xrightarrow{N} C$$

Additional Structure

Work in a *closed autonomous monoidal* bicategory

- Closed: Right-adjoints to $M \otimes_A -$ and $- \otimes_B M$ given by Source-Hom and Target-Hom
- Monoidal: $\otimes_R$ on 0-, 1-, and 2-cells; a symmetric monoidal product
- Autonomous: An involution $(-)^{op}$
  For $M \in \mathscr{M}_R(A, B)$, this gives
  - $M^{op} \in \mathscr{M}_R(B^{op}, A^{op})$
  - $M_r \in \mathscr{M}_R(A \otimes_R B^{op}, R)$
  - $M_\ell \in \mathscr{M}_R(R, A^{op} \otimes B)$
  - ...compatibility axioms

Further Examples

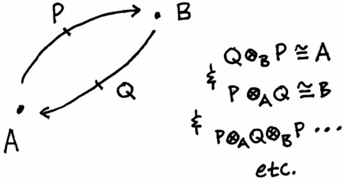
Let R be a commutative DG-algebra or ring spectrum.

Other examples of interest:

- Ch_R
  $Ch_R(A, B)$ is the category of DG-(B, A)-bimodules
- $\mathscr{D}_R$
  $\mathscr{D}_R(A, B)$ is the homotopy category of (B, A)-bimodules
  Note: Use $\otimes$ and Hom to denote the derived tensor and hom.

Generalized Morita Theory
(Baez Week # 209)

Morita equivalence is 1-cell equivalence in a bicategory.



(pre-)Azumaya objects are 0-cells, A, in a monoidal bicategory for which
$\exists B$ such that $A \otimes B \sim_M$ unit object, R.

Generalized Picard and Brauer Groups
(Baez Week # 209)

The Picard group of R is the group of self-Morita-equivalences of R.

$$\begin{tikzcd} R \arrow[loop above, "P"] \end{tikzcd}$$

if P invertible, $P \in Pic(R)$

The Brauer group of R is the group of Azumaya objects, up to Morita equivalence.

Morita Theory for Rings

Theorem (Morita)

Let R and S be rings. Then $Mod\text{-}R \simeq Mod\text{-}S$ if and only if
there exists an R-S-bimodule, ${}_RP_S$, such that:

- P_S is a finitely-generated and projective S-module.
- P_S generates $Mod\text{-}S$.

  'generator' means every S-module can be resolved by P
  e.g. S generates $Mod\text{-}S$
- $\mathrm{Hom}_S({}_RP_S, {}_RP_S) \cong {}_RR_R$ as an R-R-bimodule.

– Characterization of Morita equivalences

– Concrete description

Brauer Group of a Commutative Ring

Let R be a commutative ring, and A an R-algebra.

- A is *central* if the center of A is equal to R.
- A is *separable* if A is projective as a module over $A^e = A \otimes_R A^{op}$.
- A is *faithfully-projective* if A is finitely-generated and projective
  as an R-module, and if,
  for any R-module M, $A \otimes_R M = 0 \Rightarrow M = 0$.

Brauer Group of a Commutative Ring

Theorem

The following are equivalent for an R-algebra A:

1. *A is central and separable over R.*
2. *A is faithfully-projective over R and $\mu : A^e \xrightarrow{\cong} Hom_R(A, A)$ is an isomorphism.*
3. *A^e is Morita equivalent to R.*
4. *There is an R-algebra B such that $A \otimes_R B$ is Morita equivalent to R.*

These conditions define an *Azumaya algebra* over R. The Brauer group of R is the group of Azumaya R-algebras, up to Morita equivalence.

Duality and Invertibility

Duality in a closed bicategory generalizes duality in a closed monoidal category.

$$X : A \nrightarrow B \qquad Y : B \nrightarrow A$$

A pair of 1-cells (X, Y) is a *dual pair* if

there are maps

- unit: $B \to X \otimes_A Y$
  counit:
  $Y \otimes_B X \to A$.
  satisfying the triangle identities.

Equivalently:

- (X, Y) defines an adjunction $(- \otimes_B X) \dashv (- \otimes_A Y)$.
- (X, Y) defines an adjunction $(Y \otimes_B -) \dashv (X \otimes_A -)$.

Duality and Invertibility

A dual pair, (X, Y) is *invertible* if the induced functors are an equivalence;

if and only if the unit/counit are isomorphisms

Remark: (X, Y) is an invertible pair if and only if (Y, X) is invertible.

Characterization of Standard Equivalences

Let $\mathscr{B}$ be a closed autonomous monoidal bicategory with unit R, and let $F\colon \mathscr{B}(A,R) \xrightarrow{\simeq} \mathscr{B}(B,R)$ be an equivalence of categories.

Theorem (Johnson)

The functor F is naturally isomorphic to a functor of the form $- \otimes_A P$
$\iff$ F is a component of an invertible strong transformation.
$\iff$ F is a component of an invertible Hom-*enriched transformation.*

Invertible Strong Transformations

A family of equivalences

$$F_c \colon \mathscr{B}(A, C) \xrightarrow{\simeq} \mathscr{B}(B, C) \qquad \text{for all } C$$

such that $K \otimes_C F_C(X) \xrightarrow{\cong} F_{C'}(K \otimes_C X)$ naturally for all K, X.

$$\begin{tikzcd} C \arrow[d, "K"'] \\ C' \end{tikzcd} \qquad \begin{tikzcd} \mathscr{B}(A,C) \arrow[r, "F_C"] \arrow[d, "K\otimes_C -"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow}"] & \mathscr{B}(B,C) \arrow[d, "K\otimes_C -"] \\ \mathscr{B}(A,C') \arrow[r, "F_{C'}"] & \mathscr{B}(B,C') \end{tikzcd}$$

Proof: bicategorical Yoneda lemma.

Invertible Hom-enriched Transformations

A family of enriched equivalences

$$F_c \colon \mathscr{B}(A, C) \xrightarrow{\simeq} \mathscr{B}(B, C) \qquad \text{for all } C$$

such that, for all 1-cells T, U, V with common source, A,

$$\begin{tikzcd}
\mathrm{Hom}_A(U,V) \otimes_{\bullet} \mathrm{Hom}_A(T,U) \arrow[r] \arrow[d] & \mathrm{Hom}_A(T,V) \arrow[d] \\
\mathrm{Hom}_A(FU,FV) \otimes_{\bullet} \mathrm{Hom}_A(FT,FU) \arrow[r] & \mathrm{Hom}_A(FT,FV)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_A(T,T) \arrow[r] & \mathrm{Hom}_B(FT,FT) \\
\left(\begin{smallmatrix}\text{target}\\ \text{of } T\end{smallmatrix}\right) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Proof: enriched category theory.

Characterization of Standard Equivalences

Let $\mathscr{B}$ be a closed autonomous monoidal bicategory with unit R, and let $F\colon \mathscr{B}(A, R) \xrightarrow{\simeq} \mathscr{B}(B, R)$ be an equivalence of categories.

Theorem (Johnson)

The functor F is naturally isomorphic to a functor of the form $- \otimes_A P$
$\iff$ F is a component of an invertible strong transformation.
$\iff$ F is a component of an invertible Hom*-enriched transformation.*

Characterization of Azumaya Objects

Theorem (Johnson)

The following are equivalent for a 0-cell, A, with $A_r \colon A^e \to R$:

- *A_r is an invertible 1-cell.*
- *A_r is dualizable over R and there is a 0-cell B such that $A \otimes_R B \sim_M R$. (Morita equivalence)*

If $\mathscr{B}$ is triangulated, these are equivalent to the following:

- *A_r is dualizable over R*
  and $A^e \cong \mathrm{End}_R(A_r)$
  and R is $A_r^{\otimes}$-local.

We will soon explain the words *triangulated* and *local* in this context.

Characterization of Azumaya Objects

A_r is invertible if and only if there is a 0-cell B such that B_r is left-dualizable and $A \otimes_R B \simeq_{Morita} R$.

Proof: Diagram chase

$$
\begin{tikzcd}
\mathscr{D}(R,-) \arrow[r, shift left, "-\otimes_R A_r"] \arrow[dr, shift left] & \mathscr{D}(A^e,-) \arrow[l, shift left, "\mathrm{Hom}_{A^e}(A_r,-)"] \arrow[d, shift left, "-\otimes_{A^e}(A^e\otimes_R B_r)"] \\
& \mathscr{D}(A^e \otimes_R B^e,-) \arrow[u, shift left, "\mathrm{Hom}"] \arrow[ul, shift left, "\simeq"]
\end{tikzcd}
$$

If B_r is left-dualizable, the vertical counit is an isomorphism. (Lemma)

Question: does $A_r \otimes_R B_r$ invertible $\implies$ A_r, B_r each dualizable?

Triangulated Bicategories

Let $\mathscr{D}$ be a closed autonomous monoidal bicategory, and each $\mathscr{D}(A, B)$ is a triangulated category such that:

- The functors $X \otimes_A -$, $- \otimes_B X$, $\mathrm{Hom}_A(X, -)$, $Hom_B(X, -)$ are exact.
- axioms relating Σ and units, autonomous structure …

For the remainder of the talk, we suppose $\mathscr{D}$ has such a triangulated structure.

$\mathscr{D}[Z, W]_*$ denotes the graded Abelian group of 2-cells $Z \to W$.

Localization (skip this slide)

In a Triangulated Bicategory

Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}$.

Definition (T-acyclic)

M is *$T_\otimes$-acyclic* if $T \otimes_A M = 0$. (push-forward)

M' is *$T^\otimes$-acyclic* if $M' \otimes_B T = 0$. (pull-back)

Definition (T-local)

N is *$T_\otimes$-local* if $\mathscr{D}[M, N]_* = 0$ for all $T_\otimes$-acyclic M.

N' is *$T^\otimes$-local* if $\mathscr{D}[M', N']_* = 0$ for all $T^\otimes$-acyclic M'.

Notation:

The subcategory of $T_\otimes$-local 1-cells $C \nrightarrow A$ is $\mathscr{D}(C, A)_{\langle T_\otimes \rangle}$

The subcategory of $T^\otimes$-local 1-cells $B \nrightarrow C$ is $\mathscr{D}(B, C)_{\langle T^\otimes \rangle}$

Baker-Lazarev Factorization

The adjunctions induced by $T : A \nrightarrow B$ factor through the T-local categories [Baker-Lazarev 2004]

$$
\begin{tikzcd}
\mathscr{D}(B,-) \arrow[rr, shift left, "-\otimes_B T"] \arrow[dr, shift left, "\text{localization}"] & & \mathscr{D}(A,-) \arrow[ll, shift left, "\mathrm{Hom}_A(T,-)"] \arrow[dl, shift left, "\mathrm{Hom}_A(T,-)"] \\
& \mathscr{D}(B,-)_{\langle T^{\otimes} \rangle} \arrow[ul, shift left] \arrow[ur, shift left, "-\otimes_B T"] &
\end{tikzcd}
$$

Proposition (Baker-Lazarev 2004)

If T is left-dualizable and the action $A \xrightarrow{\cong} \mathrm{Hom}_B(T,T)$ is an isomorphism, then $\mathscr{D}(B,-)_{\langle T^{\otimes} \rangle} \simeq \mathscr{D}(A,-)$. Likewise for right-dualizability.

Summary

Morita theory and invertibility in bicategories.

- Conceptual unification of algebraic and topological theory
- Description of invertibility generalizes classical work with Azumaya algebras
- Triangulated bicategories: Factorization relates localization and invertibility

Thank you!

Morita theory and Azumaya objects in bicategorical contexts

Niles Johnson

University of Chicago

http://www.math.uchicago.edu/~niles

January 2009
Joint Meetings, Washington, D.C.

Appendix

This material was not presented during the talk, but may be of interest. We give two lemmas on duality which are especially useful. One is widely known, but the other does not seem to be; both are easy.

We also give two corollaries, both from previous work in Morita theory.

Duality and Invertibility

Lemmas

- X is right-dualizable if and only if the coevaluation $X \otimes_A \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_A(X, X)$ is an isomorphism.
  Any right dual of X is isomorphic to $\mathrm{Hom}_A(X, A)$.
- X is left-dualizable if and only if the coevaluation $\mathrm{Hom}_B(X, B) \otimes X \to \mathrm{Hom}_B(X, X)$ is an isomorphism.
  Any left dual of X is isomorphic to $\mathrm{Hom}_B(X, B)$.

Duality and Invertibility

Lemmas

- If X is right-dualizable and the action map $B \longrightarrow \mathrm{Hom}_A(X, X)$ is an isomorphism, then the evaluation $X \otimes_A \mathrm{Hom}_B(X, B) \longrightarrow B$ is an isomorphism.
- If X is left-dualizable and the action map $A \longrightarrow \mathrm{Hom}_B(X, X)$ is an isomorphism, then the evaluation $\mathrm{Hom}_A(X, A) \otimes_B X \longrightarrow A$ is an isomorphism.

Corollaries

Practical Applications

Let A be a 0-cell of $\mathscr{D}$ and take $T = A_r : A^e \nrightarrow R$.

Baker-Lazarev: $\mathscr{D}$ = spectra

Corollary

The following are equivalent:

0. A_r is Azumaya (as defined previously)
1. A_r is right-dualizable
   action: $R \xrightarrow{\cong} \mathrm{Hom}_{A^e}(A_r, A_r) = THH_R(A, A)$
   A^e is $A_{r\otimes}$-local
2. A_r is left-dualizable
   action: $A^e \xrightarrow{\cong} \mathrm{Hom}_R(A_r, A_r)$
   R is ${A_r}^{\otimes}$-local

Example: Morava $K(1)$ is Azumaya over $\widehat{KU}_2$ (Baker-Lazarev).

Corollaries
Practical Applications

Let R be a commutative differential graded algebra (Rickard)
or a commutative ring spectrum (Schwede-Shipley).
Recall: $\mathscr{D}_R(A, B)$ is the homotopy category of (B, A)-bimodules.

Corollary

Let $T : A \nrightarrow R$ be a 1-cell of $\mathscr{D}_R$, and let $E = \mathrm{Hom}_A(T, T)$.
Let $\widetilde{T}$ be the induced 1-cell $A \nrightarrow E$.
If T has the following two properties, then $\widetilde{T}$ provides an equivalence
$\mathscr{D}_R(A) \simeq \mathscr{D}_R(E)$.

- T is right-dualizable
- T generates the triangulated category $\mathscr{D}_R(A)$.

$\mathscr{D}_R(A) = \mathscr{D}_R(A, R)$; $\mathscr{D}_R(E) = \mathscr{D}_R(E, R)$

$$\begin{array}{lcl} T \text{ right-dualizable} & \Rightarrow & \widetilde{T} \text{ is right-dualizable} \\ T \text{ generates } \mathscr{D}_R(A) & \Rightarrow & A \text{ is } T_\otimes\text{-local.} \end{array}$$

For general $\mathscr{D}$, need to know E and $\widetilde{T} : A \nrightarrow E$ exist.