RESEARCH STATEMENT

NILES JOHNSON

The unifying theme of my research is mathematics that transforms abstract questions into concrete problems approachable by direct calculational tools. My fields are algebra and topology, and my interests range broadly from algebraic topology to categorical algebra to representation theory.

My core research program is on the **Brauer theory of generalized Galois extensions** in stable homotopy theory. My work in categorical algebra develops a definition and characterization theorem for the elements of the Brauer group, **Azumaya objects, in bicategorical contexts**. Current work in progress applies this to study the relative Brauer group in a Galois extension of commutative ring spectra. Calculations of norm operations are of particular interest in that project. A second project joint with Nick Gurski, Peter May, and Angélica Osorno relates the categorical unit, Picard, and Brauer groups to the **classification of stable two-types** and their Postnikov invariants.

My work on power operations with Justin Noel provides explicit computations of a norm operation in stable homotopy to address a long-standing conjecture about compatibility of **power operations on complex cobordism** with a p-local summand known as the Brown-Peterson spectrum. A second project in progress develops an obstruction theory for **rigidifying up-to-homotopy algebra maps** of spectra.

In two projects with the UGA VIGRE Algebra group we use scheme-theoretic and Lie-algebraic techniques to compute **low-degree cohomolgy for finite groups of Lie type** in positive characteristic. And a recent undertaking with John Drake uses a topological point of view to develop mathematical models for the concept of ecological niche. We also apply the techniques of topological data analysis to address empirical questions about **niche topology**.

My work in each of these areas is discussed below, together with an overview of work in progress and future plans in each area. I also give a one-page description of the undergraduate research projects I've mentored in Section 4.

1. Algebraic Topology

1.1. Calculations for Complex-Oriented Ring Spectra.

Complex oriented cohomology theories have a deep connection with the theory of formal group laws, and this provides both conceptual and computational access to stable homotopy. One seminal example is the Adams-Novikov spectral sequence converging to the stable homotpy groups of spheres [Rav03]. The cohomology theory of complex cobordism is represented by the spectrum MU, and this is the universal complex oriented cohomology theory. It is a highly-structured ring spectrum and its graded ring of homotopy groups MU_* carries the universal formal group law. A spectrum E is complex oriented when it has an orientation map $MU \to E$, and such a map always endows the graded ring E_* with a formal group law; one can use this algebraic structure to study aspects of the spectrum-level map $MU \to E$.

This is of particular interest in the p-local case; the p-local complex cobordism spectrum $MU_{(p)}$ splits as a wedge of suspensions of the Brown-Peterson spectrum BP. The graded homotopy ring

Date: November, 2011

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BP_* is an algebra over $MU_{(p)*}$, and the question of whether this lifts to a highly-structured $MU_{(p)}$-algebra structure for BP has been of particular interest throughout the study of highly-structured ring spectra (see, *e.g.*, [May75, BMMS86, Ric06]).

In a joint project, Justin Noel and I [JN10] developed computer calculations to study algebraic conditions on whether such a map is compatible with even-degree power operations, following Quillen [Qui71] and McClure [BMMS86, VIII]. By producing non-trivial obstructions, in the form of certain power-series coefficients, we were able to show for $p \leq 13$ that the algebra structure present on the level of homotopy does not rigidify to an $MU_{(p)}$-algebra structure on BP. This follows from a more general result regarding p-typical orientations:

Theorem 1.2 ([JN10]). *Let p be a prime, and suppose $f\colon MU_{(p)} \to E$ is p-local map of H_∞ ring spectra satisfying:*

i. f factors through Quillen's map to BP.
ii. f induces a Landweber exact MU_*-module structure on E_*.
iii. **Small Prime Condition:** $p \in \{2, 3, 5, 7, 11, 13\}$.

then $\pi_* E$ is a $\mathbb{Q}$-algebra.

Corollary 1.3 ([JN10]). *Suppose the Small Prime Condition holds and $n \geq 1$. The standard p-typical orientations on E_n, $E(n)$, $BP\langle n\rangle$, and BP do not respect power operations. In particular, the corresponding MU-ring structures do not rigidify to commutative MU-algebra structures.*

These results are expected to hold also for $p > 13$; the Small Prime Condition is a practical requirement imposed only for the computer calculations. Indeed, preliminary results of an undergraduate project extend these results to $p \leq 23$; that work is described in Section 4.3.

1.4. **Obstruction theory for homotopical algebra.** This project, also joint with Justin Noel, arose from our interest in determining how closely the category of H_∞ algebras approximates the derived category of E_∞ algebras. Intuitively, there ought to be a significant gap between these two concepts: the first concerns algebras in a derived category and the second concerns a derived category of algebras. However, there were no examples demonstrating this gap in homotopy theory. In this work we provide an obstruction theory and examples demonstrating the gap between E_∞ maps and H_∞ maps.

The category of E_∞ algebras is isomorphic to the category of algebras over a particular monad (triple) $\mathbb{P}$, and our approach addresses the more general question:

Question 1.5. Let T be some nice monad acting on a topologically enriched model category $\mathscr{C}$ and suppose A and B are T-algebras. Suppose we have a diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array} \tag{1.6}$$

that commutes in $ho\mathscr{C}$.

Is there a representative of the homotopy class of f such that the corresponding diagram commutes in $\mathscr{C}$? In other words, can f be rigidified to a map of T-algebras? If so, is this lift in some sense unique?

We develop an obstruction theory, in the form of a fringed Bousfield-Kan spectral sequence, which completely resolves these questions. Let $(ET)_\infty$ denote the homotopy category of strict T-algebras in $\mathscr{C}$ and let $(HT)_\infty$ denote the category of T-algebras in $Ho\mathscr{C}$. The fullness and faithfulness of the forgetful functor

$$(ET)_\infty \to (HT)_\infty$$

is analyzed by the obstruction spectral sequence $E_r^{s,t}$; we have the following two summary results:

Theorem 1.7 (Johnson-Noel). *The forgetful functor $(ET)_\infty \to (HT)_\infty$ is faithful if and only if $E_\infty^{t,t} = 0$ for $t > 0$.*

Theorem 1.8 (Johnson-Noel). *The forgetful functor $(ET)_\infty \to (HT)_\infty$ is full if and only if the differential d_r on $E_r^{0,0}$ is trivial for all $r \geq 2$.*

A feature of this work we particularly enjoy is that it explicitly resolves the subtle difference between $(ET)_\infty$ and $(HT)_\infty$, so we are able to give elementary but nontrivial examples in rational stable homotopy. For these, we consider the classical case T is the E_∞ monad $\mathbb{P}$.

For any space X, we have a mapping spectrum $H\mathbb{Q}^X$ into the Eilenberg-Mac Lane spectrum of the rationals. These spectra can be constructed directly from the rational cochains on X, and they inherit an E_∞ structure from the cup product. These spectra have the property that their homotopy groups are the rational homology of the space X:

$$\pi_{-*}(H\mathbb{Q}^X) \cong H^*(X;\mathbb{Q}).$$

Moreover, the rational cochain maps which induce commutative algebra maps on the level of cohomology correspond under this identification to the H_∞ maps of spectra:

$$H_\infty(H\mathbb{Q}^X, H\mathbb{Q}^Y) \cong \mathcal{C}omm\ \mathbb{Q}\text{-}alg\ (H^*(Y;\mathbb{Q}), H^*(X;\mathbb{Q}))\,. \tag{1.9}$$

Example 1.10. Let $X = S^2$ and $Y = S^3$. The obstruction spectral sequence collapses at E_2 and for degree reasons the E_∞ map induced by the Hopf map $\eta\colon S^3 \to S^2$ is in the kernel of the edge map. This provides an example where the forgetful functor spectra fails to be faithful and proves the following:

Theorem. *The Hopf map induces a nontrivial E_∞ map $H\mathbb{Q}^{S^2} \to H\mathbb{Q}^{S^3}$ which is trivial as an H_∞ map.*

Example 1.11. Let $X = N$ be the Heisenberg nilmanifold, the quotient of the group of uni-upper triangular 3×3 real matrices by the maximal subgroup with all integer entries. A computation with the Serre spectral sequence shows $H^*(N;\mathbb{Q})$ is generated by exterior classes x and y in degree 1, polynomial classes α and β in degree 2, with the relations

$$xy = \alpha^2 = \beta^2 = \alpha\beta = x\alpha = y\beta = x\beta + y\alpha = 0.$$

It follows, moreover, from Eq. (1.9) that the H_∞ maps $HQ^N \to HQ^{S^2}$ form a rank-two $\mathbb{Q}$-vector space generated by the dual classes δ_α and δ_β. However, a calculation of Massey products shows that there are non-trivial differentials on δ_α and δ_β in the obstruction spectral sequence. This provides an example where the forgetful functor fails to be full and proves the following:

Theorem. *The H_∞ maps δ_α and δ_β are not in the image of the forgetful functor $E_\infty \to H_\infty$ and thus do not rigidify to E_∞ maps.*

1.12. **Future Work: Norm Operations in Galois Extensions.** John Greenlees and Peter May introduced norm operations for equivariant stable homotopy theory in [GM97b]. They use norm operations in equivariant cohomology to prove analogs of the Atiyah-Segal completion theorem for module spectra over the ring spectrum of complex cobordism, MU. In recent work on the Kervaire invariant problem, Michael Hill, Michael Hopkins, and Douglas Ravenel [HHR10] define a spectrum-level norm functor. For a subgroup $H \subset G$ of finite index n, $norm_H^G$ is a functor from the category of H-spectra to the category of G-spectra. For an H-spectrum X,

$$norm_H^G(X) = \bigwedge_{H_i} (H_i)_+ \wedge_H X,$$

where the H_i form a complete set of cosets of H in G. The action of G is induced by the natural inclusion of G into the wreath product $\Sigma_n \int H$. This is in direct parallel with the definition of the Evens norm for group-cohomology [Eve91]. This topological norm satisfies the same basic properties as the more familiar algebraic norm, including multiplicativity (with respect to the smash product) and the ever-important double coset formula [HHR10], [GM97b].

In the case that G is cyclic of prime order p, and H is the trivial group, the topological norm is homotopy equivalent to the extended power construction on a (non-equivariant) spectrum X:

$$norm_1^{\mathbb{Z}/p} X \simeq D_{\mathbb{Z}/p} X = E\mathbb{Z}/p \ltimes_{\mathbb{Z}/p} X.$$

This extended power construction is the basic ingredient in the study of power operations for cohomology theories; see for example [tD68], [Qui71], and [BMMS86]. The software developed to compute power operations for [JN10] can be extended to compute norm operations, and one area for future work is applying these calculations to expand our understanding of Galois theory for highly-structured ring spectra.

1.13. **Ecological Niche Topology.** During the Summer of 2011 I began talking with John Drake in the UGA Ecology department about ecological niches. The concept of a niche is a way that ecologists abstract a species' environmental needs from its geographic location. This helps ecologists address questions about large-scale population dynamics such as predicting invasive behavior or the effects of climate change. The niche is described as the set of environmental conditions in which a population can persist, and is usually thought of as a convex subset of the space of relevant environmental parameters (temperature, precipitation, soil pH, etc.).

There are number of competing attempts to make the niche concept precise, and a dearth of empirical results against which these attempts can be evaluated. In particular, the convexity assumption is rarely treated in ecological experiment or theory. I joined John's lab to work on mathematical models for the concept of a niche, with the goal of producing both theoretical and empirical conditions which will help to differentiate the various niche theories. Currently we are combining methods of machine learning—Support Vector Data Descriptors (SVDD) [TD04]—with homology calculations to answer questions about data collected in the field.

Question 1.14. Let N be the niche of a given species. Is N connected? Simply-connected? Convex? These same questions can be asked of an empirical approximation, $\overline{N}$, derived from experimental data.

Applied to data collected in the field, the machine learning techniques return $\overline{N}$ as a union of balls. The homology of such a region is eminently calculable, and our results will provide the first empirical answers to the questions above.

2. Categorical Algebra

2.1. **Homotopical Brauer Theory.** This work develops foundations for Brauer theory in homotopical settings. We consider Azumaya objects in closed autonomous monoidal bicategories, and in particular focus on the triangulated bicategories arising as homotopy bicategories of rings and ring spectra. (The term "closed" refers to the existence of internal homs, and "autonomous" refers to opposites such as opposite algebras.)

Now we state the main definitions and results.

Definition 2.2 (Eilenberg-Watts Equivalence)**.** Let A and B be a 0-cells of a bicategory $\mathscr{B}$. We say A is Eilenberg-Watts equivalent to B if and only if there exists an invertible 1-cell $T : A \nrightarrow B$.

Definition 2.3 (Brauer Group, Azumaya Objects)**.** Let $\mathscr{B}$ be a closed autonomous monoidal bicategory with unit 0-cell k. The *Brauer group* of $\mathscr{B}$, denoted $Br(\mathscr{B})$, is the group of 1-cell-equivalence-classes (Eilenberg-Watts equivalence classes) of 0-cells A for which there exists a 0-cell B such that $A \otimes_k B$ is Eilenberg-Watts equivalent to k. Such 0-cells are called *Azumaya objects*.

We give a general characterization theorems of Azumaya objects which generalizes the classical notion for fields or commutative rings the classical characterization theorem, making computational work with these objects possible. For example, we have the following:

Theorem 2.4 ([Joh10]). *Let k be a commutative ring spectrum, and A a k-algebra which is cofibrant as a k-module. Then A is Azumaya over k if and only if A is faithfully projective over k and the canonical map*

$$A \wedge_k A^{\mathsf{op}} \to F_k(A, A)$$

is a weak equivalence, where F denotes the mapping spectrum of right k-modules.

We show, moreover, that the homotopical notion of Azumaya we develop coincides with those of Toën [Toë10] for derived categories of rings and Baker-Richter-Szymik [BRS10] for commutative ring spectra. Functoriality of our general approach allows us to prove the following:

Theorem 2.5 ([Joh10]). *Let k be a commutative ring. The Eilenberg-Mac Lane functor to symmetric spectra*

$$H\colon \mathscr{C}_k \to \mathscr{C}_{Hk}$$

induces an isomorphism of Brauer groups

$$H\colon Br(\mathscr{D}_k) \to Br(\mathscr{D}_{Hk}).$$

Corollary 2.6 ([Joh10]). *If k is a field, then $Br(\mathscr{D}_{Hk}) \cong Br(k)$. In particular, $Br(\mathscr{D}_{Hk}) = 0$ if k is finite or algebraically closed.*

2.7. **Future Work: Brauer Groups and Galois Theory.** Brauer groups are an established area of interest in algebra and algebraic geometry, but Brauer theory in homotopical settings is still an emerging field. My research program focuses in particular on the relative Brauer groups of Galois extensions. This provides a source for new examples of Azumaya objects (crossed-product algebras), and generalizes to derived settings the classical connection between Brauer and Galois Theory.

Chase, Harrison, and Rosenberg [CHR65] established this connection in the case of commutative rings, and one of their main results is the following:

Theorem 2.8 (Chase-Rosenberg exact sequence). *If $R \to T$ is a Galois extension of commutative rings with Galois group G, then there is a natural exact sequence*

$$1 \to H^1(G, T^\times) \to Pic(R) \to Pic(T)^G \to H^2(G, T^\times) \to Br(T/R) \to H^1(G, Pic(T)).$$

In this theorem, $Pic(T)^G$ denotes the subgroup of the Picard group $Pic(T)$ invariant under the induced action of G. Two important corollaries illustrate the pivotal role of Brauer/Galois theory for commutative rings. Recall, for the case of fields, that the Picard group of a field is always trivial.

Corollary 2.9 (Hilbert's Theorem 90). *If $Pic(R) = 1$, then $H^1(G, T^\times) = 1$.*

Corollary 2.10 (Crossed Product Theorem). *If $Pic(T) = 1$, then $Br(T/R) = H^2(G, T^\times)$.*

I'm currently working on a project studying Amitsur cohomology for spectra, with one goal being a generalized version of the Chase-Rosenberg sequence for homotopical calculations. This would enable one to compute Brauer groups and Picard groups for Galois extensions of ring spectra. In work with Angélica Osorno, described below, we recover a similar exact sequence from categorical models of stable one-types. A key feature of that work is identifying the relative term via a cokernel construction.

2.11. **Stable n-types.** This is a joint project with Nick Gurski, Peter May, and Angélica Osorno studying categorical models for stable n-types, mainly for $n = 1$ and 2. Angélica and I have initial results for $n = 1$, and the four of us are working on $n > 1$. This has close ties with earlier work of Conduché [Con84], Joyal-Street [JS93], Garzón-Miranda [GM97a], and many others. With this classification we relate the Postnikov invariants of a stable n-type to algebraic data on the categorical side. This fits into a long exact sequence of homotopy groups paralleling that on the topological side.

Definition 2.12. A *symmetric Picard groupoid* $\mathscr{C}$ is a symmetric monoidal groupoid such that every object is invertible. The category $\mathcal{SymPic}$ has as objects the symmetric Picard groupoids and as morphisms symmetric monoidal functors. A symmetric monoidal functor is a *weak equivalence* if it is an equivalence of the underlying categories. The homotopy groups $\pi_0\mathscr{C}$ and $\pi_1\mathscr{C}$ are, respectively, the group of isomorphism classes of objects and the group of automorphisms of the unit object.

In the special case that $\mathscr{C}$ is the Picard groupoid whose objects are Azumaya k-algebras and morphisms are isomorphism classes of bimodules, $\pi_0\mathscr{C}$ and $\pi_1\mathscr{C}$ are, respectively, the Brauer and Picard groups of k. We show that the homotopy of symmetric Picard groupoids models the homotopy of stable one-types, and use this to obtain algebraic models for Postnikov systems of such. Two of our results are the following:

Theorem 2.13 (Johnson-Osorno)**.** *The stable one-types with $\pi_0 = G$ and $\pi_1 = M$ are classified by the symmetric structures on a strict and skeletal Picard groupoid with objects G and each endomorphism group isomorphic to M.*

Theorem 2.14 (Johnson-Osorno)**.** *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a map of Picard groupoids. There is a symmetric monoidal bicategory* $\operatorname{Coker}(F)$ *with a universal map $\mathscr{D} \to \operatorname{Coker}(F)$ which gives rise to a long exact sequence of homotopy groups*

$$0 \to \pi_2 \operatorname{Coker}(F) \to \pi_1\mathscr{C} \to \pi_1\mathscr{D} \to \pi_1 \operatorname{Coker}(F) \to \pi_0\mathscr{C} \to \pi_0\mathscr{D} \to \pi_0 \operatorname{Coker}(F) \to 0.$$

3. Representation Theory

During my time at UGA I became involved in the VIGRE[1] Algebra research group. This is a large research group including graduate students, postdocs, and senior faculty. The group has been working for several years on projects in representation theory of finite groups of Lie type; I participated in their sixth and seventh terms, and we wrote two papers during that time: [UGA11a, UGA11b].

We work over an algebraically closed field k of characteristic p, and let G be a simple, simply-connected algebraic group over $\mathbb{F}_p$. Our results are in the form of specific low-degree cohomology calculations via comparison between the cohomology of a finite group of Lie type, $G(\mathbb{F}_q)$, and the cohomology of the corresponding algebraic group, G. These calculations significantly extend—and provide new proofs for—earlier results of Cline, Parshall, Scott, [CPS75], Jones [Jon75], and Jones-Parshall [JP76] who considered some special cases of our coefficient modules. Here are two representative results:

Theorem 3.1 ([UGA11b])**.** *Let p be a prime, $q = p^r$, and assume $p > 3$ if G is of classical type or $p > 5$ if G is of exceptional type. Suppose $\lambda \le \omega_j$ for some j, and suppose the* Weight Condition *(defined below) holds for λ. Then the restriction map*

$$H^t(G, L(\lambda)) \to H^t(G(\mathbb{F}_q), L(\lambda))$$

is an isomorphism for $t = 1, 2$.

The Weight Condition is only needed for the degree-two isomorphism, and holds when λ is neither the highest short root nor the highest long root (except for the case $G = A_2$, with $q = 5$.) It is a technical but approachable condition:

Definition 3.2 (Weight Condition)**.** We say that $\lambda \in X(T)_+$ satisfies the *Weight Condition* if

$$\max\left\{-(\nu, \gamma^\vee) \middle| \gamma \in \Delta, \nu \text{ a weight of } \operatorname{Ext}^1_{U_r}(k, L(\lambda))\right\} < q.$$

We also have results for the finite group cohomology, using a calculation of the algebraic group cohomology and the comparison Theorem 3.1:

[1]VIGRE is an NSF program promoting the Vertical InteGration of Research and Education.

Theorem 3.3 ([UGA11b])**.** *Under the assumptions above, $H^2(G(\mathbb{F}_q), L(\lambda)) = 0$ except possibly seven cases in types E_7 and E_8, and infinitely many cases in type C_n when $\lambda = \omega_{2i}$ and $p \le n$.*

The seven exceptional cases are unknown only for specific primes $5, 7, 31$. We give an analysis of the remaining interesting case, $G = C_n = Sp_{2n}$ and $\lambda = \omega_{2i}$, with a combinatorial description of Weyl module composition factors due to Adamovich and communicated by Kleshchev-Sheth [KS99, KS01]. Using using software I developed for this purpose [Joh11], we have a number of interesting non-vanishing results for degree-two cohomology of C_n. These are listed in Figures 1 and 2. The reader will note that there is a tantalizing base-p pattern to the results. We have yet to formulate a general theorem, but this is the subject of ongoing work of myself and Chris Drupieski.

n	j
6	6
7	6
8	*none*
9	6
10	6
11	*none*
12	6
13	6
14	*none*

n	j
15	6, 8
16	6, 10
17	*none*
18	6, 14
19	6, 16
20	18
21	6, 18
22	6, 18
23	18

n	j
24	6, 8, 18
25	6, 10, 18
26	*none*
27	6, 14
28	6, 16
29	18
30	6, 18
31	6, 18
32	18

n	j
33	6, 8, 18
34	6, 10, 18
35	*none*
36	6, 14
37	6, 16
38	18
39	6, 18, 20

Figure 1. ($p = 3$) All values of n and j for which $H^2(Sp_{2n}(\mathbb{F}_q), L(\omega_j)) \ncong 0$ for $n < 40$. In each case, H^2 is 1-dimensional.

n	j
10	10
11	10
12	10
13	10
14	*none*
15	10
16	10
17	10
18	10
19	*none*

n	j
20	10
21	10
22	10
23	10
24	*none*
25	10
26	10
27	10
28	10
29	*none*

n	j
30	10
31	10
32	10
33	10
34	*none*
35	10, 12
36	10, 14
37	10, 16
38	10, 18
39	*none*

n	j
40	10, 22
41	10, 24
42	10, 26
43	10, 28
44	*none*
45	10, 32
46	10, 34
47	10, 36
48	10, 38
49	*none*

n	j
50	10, 42
51	10, 44
52	10, 46
53	10, 48
54	50

Figure 2. ($p = 5$) All values of n and j for which $H^2(Sp_{2n}(\mathbb{F}_q), L(\omega_j)) \ncong 0$ for $n < 55$. In each case, H^2 is 1-dimensional.

4. UNDERGRADUATE RESEARCH

My undergraduate mentoring is motivated by my research programs, but my main goals are to let students experience mathematics freely while guiding them toward a piece of work they can be proud of. I've been involved with two undergraduate research projects, summarized below.

4.1. **Cross-polytope numbers.** One of my students, Jack Farnsworth, worked on a combinatorial problem related to the number of paths of given length along the integer lattice in $\mathbb{R}^d$. He related this to the cross-polytope numbers in dimension d, $\diamond_d(n)$, defined in [Kim03] as a generalization of Euler's polygonal numbers. Here are the low-dimensional cases:

$$\diamond_1(n) = n, \qquad \diamond_2(n) = n^2, \qquad \diamond_3(n) = \frac{2}{3}n^3 + \frac{1}{3}n.$$

Jack was an extremely motivated student and almost entirely self-directed. My role was mainly to help him find mathematical expression for the ideas and questions he was considering. His main theorem uses a recursive formula for the path-counting problem to give a recursive formula for the cross-polytope numbers:

Theorem 4.2. $\diamond_d(n) = \diamond_{d-1}(n) + \diamond_d(n-1) + \diamond_{d-1}(n-1)$

4.3. **Complexity of the p-typical formal sum.** Another student, Eddie Beck, wanted to work on something related to the formal group law computations in my previous work. He developed complexity estimates for the algorithms, giving a detailed analysis of the p-typical formal group law which has resulted in a number of substantial improvements. His preliminary results extend the range of primes for which the results of [JN10] hold:

Theorem 4.4. *Quillen's map $MU \to BP$ does not carry H_∞ structure for $p = 17, 19, 23$.*

By discovering a connection with specialized partitions called *Mahler partitions* [Mah81], Eddie was able to give a parallelizable algorithm for the formal sum.

Definition 4.5. Let n and p be positive integers. A *base-p Mahler partition* of n is a sequence $k = (k_0, k_1, \ldots)$ such that

$$n = \sum k_i p^i$$

for nonnegative integers k_i. (Note that there is no restriction on the size of k_i relative to p.) The *length* of a partition is $|k| = \sum_i k_i$. If k is a Mahler partition of n (working implicitly at a fixed base p), we write $k \rhd n$.

Definition 4.6. Let p be a prime. We define, respectively, the coefficients of the universal p-typical logarithm, exponential, and formal sum as follows:

$$\log(t) = \sum \ell_i t^{p^i} \qquad F(x,y) = \exp(\log(x) + \log(y))$$

$$\exp(t) = \log^{-1}(t) = \sum_i e_i t^i \qquad = \sum_i e_i (\log(x) + \log(y))^i$$

Theorem 4.7. *Let $c(n)$ be the term of total-degree n in $F(x,y)$, so $F(x,y) = \sum_n c(n)$. Then*

$$c(n) = \sum_{k \rhd n} f(k)$$

where the sum is over all base-p Mahler partions of n and

$$f(k) = e_{|k|} \cdot \prod_i \ell_i^{k_i} (x^{p^i} + y^{p^i})^{k_i} \cdot \frac{|k|!}{\prod k_i!}.$$

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