Modeling stable One-types

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January 7, 2012

joint with Niles Johnson

n-types

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A lot of work has been done in this field:

- Whitehead: groups (connected 1-types), crossed modules (connected 2-types)
- Loday: cat-n-groups
- Conduché: crossed modules of length two
- Baues: identifying specific data

Homotopy hypothesis

Grothendieck's homotopy hypothesis

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For $n = 1$, there is an equivalence of homotopy categories:

$$B : \mathrm{Ho}(Gpd) \rightleftarrows \mathrm{Ho}(Top_{\leqslant 1}) : \Pi_1.$$

Stable homotopy n-types

Definition

A *connective stable homotopy n-type* is a spectrum X such that for all $i > n$ and $i < 0$

$$\pi_i(X) = 0.$$

We denote by $\mathcal{S}_0^n$ the full subcategory of the category of spectra given by the connective stable n-types.

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Guess for an algebraic model

unstable 1-types $\leftrightarrow$ groupoids

E_∞ spaces $\leftrightarrow$ symmetric monoidal

group-like $\leftrightarrow$ invertible objects

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The functors B and Π_1 preserve the abelian groups π_0 and π_1.

The k-invariant

The remaining data for a stable 1-type X is the k-invariant

$$k_0 \in [K(\pi_0 X, 0), K(\pi_1 X, 2)]_{\mathrm{st}} \cong Hom(\pi_0 X / 2\pi_0 X, \pi_1 X),$$

which corresponds to

$$\eta^* : \pi_0 X \to \pi_1 X.$$

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- *a cocycle $h \in C^3(\pi_0\mathcal{C}, \pi_1\mathcal{C})$, defined by the associativity ismorphism;*
- *a function $c : (\pi_0\mathcal{C})^2 \to \pi_1\mathcal{C}$, defined by the symmetry isomorphism.*

Theorem

A Picard groupoid $\mathcal{C}$ is equivalent to a skeletal and strict Picard groupoid, with objects given by $\pi_0\mathcal{C}$, automorphisms given by $\pi_1\mathcal{C}$ and symmetry isomorphism defined by the quadratic map

$$\begin{aligned} \pi_0\mathcal{C} &\to \pi_1\mathcal{C}, \\ x &\mapsto c(x, x). \end{aligned}$$

The truncated sphere spectrum

Let $\mathbb{S}$ be the skeletal Picard groupoid with $ob\mathbb{S} = \mathbb{Z}$,

$$\mathbb{S}(m,n) = \begin{cases} 0 & \text{if } m \neq n \\ \mathbb{Z}/2 & \text{if } m = n, \end{cases}$$

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Let $\mathcal{E}$ be the symmetric monoidal category of finite pointed sets and isomorphisms. There is a symmetric monoidal functor

$$\mathcal{E} \to \mathbb{S}.$$

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Proof.

$$\begin{array}{ccc} B\mathcal{E} & \longrightarrow & B\mathbb{S}. \\ \downarrow & \nearrow_{\exists} & \\ \Omega B(B\mathcal{E}) & & \end{array}$$

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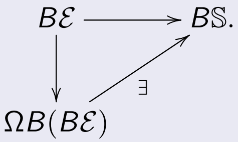
Proof.



The diagonal map is an isomorphism on π_0 and π_1. □

Note: We can also describe a bipermutative structure on $\mathbb{S}$ such that $B\mathbb{S}$ is the 1-truncation of the zeroth space of the sphere spectrum in the category of E_∞ *ring* spaces.

The homotopy cofiber

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3. The 2-cells between (f, N) and (f', N') are morphisms $\alpha : N \to N'$ of $\mathcal{C}$ such that :

$$\begin{array}{ccc} & X & \\ {\scriptstyle f} \swarrow & & \searrow {\scriptstyle f'} \\ Y \otimes F(N) & \xrightarrow[1 \otimes F(\alpha)]{} & Y \otimes F(N') \end{array}$$

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$$c_F : \mathcal{D} \to Coker(F)$$

that induces a long exact sequence

$$0 \longrightarrow \pi_2 Coker(F) \longrightarrow \pi_1\mathcal{C} \longrightarrow \pi_1\mathcal{D} \longrightarrow \pi_1 Coker(F) \longrightarrow \pi_0\mathcal{C} \longrightarrow \pi_0\mathcal{D} \longrightarrow \pi_0 Coker(F) \longrightarrow 0$$

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Corollary

The classifying space BCoker(F) is a model for the homotopy cofiber of BF.

Looking into the future

Use similar ideas to model stable2-types (n-types?) with Picard bigroupoids (n-groupoids?).