For complex oriented cohomology theories, p-typicality is atypical

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Complex Cobordism

MU is a nexus in stable homotopy theory.

- There is a spectrum *MU* satisfying:

$$\pi_n MU \cong \{\text{Complex cobordism classes of n-manifolds}\}.$$

- There is a spectral sequence (ANSS)

$$Ext^{*,*}_{MU_* MU}(MU_*, MU_*) \implies \pi_* S.$$

- *MU* serves as a conduit between the theory of formal group laws and stable homotopy theory.

This project: use power series calculations to get results about power operations in complex-oriented cohomology theories

Goal

Conjecture

The p-local Brown Peterson spectrum BP admits an E_∞ ring structure

Partial Results:

- (Basterra-Mandell) BP is E_4.
- (Richter) BP is $2(p^2 + p - 1)$ homotopy-commutative.
- (Goerss/Lazarev) BP and *many* of its derivatives are $E_1 = A_\infty$-spectra under MU (in many ways).
- (Hu-Kriz-May) There are no H_∞ ring maps $BP \to MU_{(p)}$. H_∞ is an "up to homotopy" version of E_∞

Goal

Theorem (J. – Noel)

Suppose $f : MU_{(p)} \to E$ is map of H_∞ ring spectra satisfying:

1. f factors through Quillen's map to BP.
2. f induces a Landweber exact MU_*-module structure on E_*.
3. **Small Prime Condition:** $p \in \{2, 3, 5, 7, 11, 13\}$.

then $\pi_* E$ is a $\mathbb{Q}$-algebra.

Application: The standard complex orientations on E_n, $E(n)$, $BP\langle n \rangle$, and BP do not respect power operations;
The corresponding MU-ring structures do not rigidify to commutative MU-algebra structures.

Plan

- Motivate structured ring spectra
- Describe *MU*, *BP*, and the connection to formal group laws
- Topological question $\rightsquigarrow$ algebraic question (power series)
- Display some calculations

Spectra $\leftrightarrow$ Cohomology theories

A (pre-)spectrum is a sequence of pointed spaces, E_n, with structure maps

$$\Sigma E_n \to E_{n+1}$$

such that the adjoint is a homotopy equivalence:

$$E_n \xrightarrow{\simeq} \Omega E_{n+1}.$$

This yields a reduced cohomology theory on based spaces:

$$\widetilde{E}^n(X) = [X, E_n] \cong [X, \Omega E_{n+1}] \cong \widetilde{E}^{n+1}(\Sigma X)$$

Spectra $\leftrightarrow$ Cohomology theories

Some motivating examples:

- Ordinary reduced cohomology is represented by Eilenberg-Mac Lane spaces

$$\widetilde{H}^n(X, R) = [X, K(R, n)]$$

- Topological K-theory is represented by $BU \times \mathbb{Z}$ and U (Bott periodicity):

$$\widetilde{KU}^n(X) = \begin{cases} [X, BU \times \mathbb{Z}] & n = even \\ [X, U] & n = odd \end{cases}$$

- Complex cobordism is represented by $MU(n) = \mathrm{colim}_q \Omega^q TU(n+q)$

$$\widetilde{MU}^n(X) = [X, MU(n)]$$

- etc.

Spectra $\leftrightarrow$ Cohomology theories

From a spectrum E we get an unreduced cohomology theory on unbased spaces by adding a disjoint basepoint.
For an unbased space X,

$$E^*(X) = \widetilde{E}^n(X_+) = [X_+, E_*]$$

$E^*(-)$ takes values in graded abelian groups.

When E is a ring spectrum, $E^*(-)$ takes values in graded commutative rings (with unit).
E^* denotes the graded ring $E^*(pt.)$.

Spectra $\leftrightarrow$ Cohomology theories

Brown Representibility

Every generalized cohomology theory is represented by a spectrum.

Viewed through this lens, it is desirable to express the “commutative ring” property in the category of spectra.

Doing so allows us to work with cohomology theories as algebraic objects.

Difficulty: organizing higher homotopy data
(motivates operads & monads)

There are many good categories of spectra, having well-behaved smash products and internal homs.

Structured Ring Spectra

The category of E_∞ ring spectra is one category of structured ring spectra. An E_∞ ring spectrum is equipped with a coherent family of structure maps

$$\begin{array}{ccc} E^{\wedge s} & \longrightarrow & E \\ \downarrow & \overset{\mu}{\nearrow} & \\ D_s & & \end{array}$$

which extend over the Borel construction $D_s E = E\Sigma_s \ltimes_{\Sigma_s} E^{\wedge s}$; a "homotopy-fattened" version of E^s

coherent: $D_s D_t \to D_{st}$, $D_s \wedge D_t \to D_{s+t}$, etc.

Power Operations and H_∞ Ring Spectra

- The definition of E_∞ predated applications by about 20 years
- For many applications, it suffices to have the coherent structure maps defined only in the homotopy category.
  This defines the notion of an H_∞ ring spectrum.
- This data corresponds precisely to a well-behaved family of power operations in the associated cohomology theory.

For an unbased space X, and $\pi \leq \Sigma_n$

$$P_\pi : E^0(X) \xrightarrow{\mu} E^0(D_s X) \xrightarrow{\delta^*} E^0(B\pi \times X).$$

μ: H_∞ structure maps
δ^*: pulling back along diagonal $X \to X^{\times s}$

Power Operations and H_∞ Ring Spectra

MU has a natural H_∞ ring structure arising from the group structure on BU.

Thom isomorphism for $MU \Rightarrow$ wider family of even-degree power operations

$$P_\pi : MU^{2i}(X) \to MU^{2in}(B\pi \times X)$$

$\pi \leq \Sigma_n$

take $\pi = C_p$, $X = pt$.

$MU^*(\mathbb{C}P^\infty) = MU^*[\![x]\!]$ also has a (formal) group structure induced by the multiplication on $\mathbb{C}P^\infty$. This gives us computational access to the MU power operations.

Formal Group Laws

A (commutative, 1-dimensional) formal group law over a ring R is determined by a power series $F(x,y) \in R[\![x,y]\!]$ which is unital, commutative, and associative, in the following sense:

- $F(x,0) = x = F(0,x)$.
- $F(x,y) = F(y,x)$.
- $F(F(x,y),z) = F(x,F(y,z))$.
- Example ($\mathbb{G}_a$): $F(x,y) = x + y$.
- Example ($\mathbb{G}_m$): $F(x,y) = x + y + xy$.
- Example (MU): $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ induces

$$\begin{array}{ccc} MU^*(\mathbb{C}P^\infty) & \longrightarrow & MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\ \| & & \| \\ MU^*[\![x]\!] & \longrightarrow & MU^*[\![x,y]\!] \\ & & \\ x & \longmapsto & x +_{MU} y \end{array}$$

Formal Group Laws

Theorem (Lazard)

There is a universal formal group law

$$F_{univ.}(x, y) = \sum a_{ij} x^i y^j$$

and it is defined over

$$L = \mathbb{Z}[U_1, U_2, U_3, \ldots]$$

Theorem (Quillen)

$$MU^* = \mathbb{Z}\,[U_1, U_2, U_3, \ldots]$$

and

$$x +_{MU} y = F_{univ.}(x, y)$$

MU^* and BP^*

$$MU^* = \mathbb{Z}[U_1, U_2, U_3, \ldots]$$

$$MU^{-*} \otimes \mathbb{Q} \cong H\mathbb{Q}_*(MU) \cong \mathbb{Q}[m_1, m_2, m_3, \ldots]$$

$$[\mathbb{C}P^n] \in MU^{-2n}$$

Under the Hurewicz map to rational homology

$$[\mathbb{C}P^n] \mapsto (n+1)m_n.$$

$$\mathbb{Q}\left[[\mathbb{C}P^1], [\mathbb{C}P^2], [\mathbb{C}P^3], \ldots\right] \xhookrightarrow{\cong} MU^* \otimes \mathbb{Q}$$

MU^* and BP^*

$$MU^* \xrightarrow{r_*} BP^*$$
$$MU^* \otimes \mathbb{Q} \longrightarrow BP^* \otimes \mathbb{Q}$$
$$m_i \mapsto \begin{cases} 0 & \text{if } i \neq p^k - 1 \\ \ell_k & \text{if } i = p^k - 1 \end{cases}$$

$$\mathbb{Q}[\ell_1, \ell_2, \ell_3, \ldots] \xhookrightarrow{\cong} BP^* \otimes \mathbb{Q}$$

$$r_*[\mathbb{C}P^{p^k - 1}] = p^k \ell_k \in BP^{-2(p^k-1)} \otimes \mathbb{Q}$$
$$r_*[\mathbb{C}P^n] = 0 \quad \text{for } n \neq p^k - 1$$

MU^* and BP^*

$$MU = \bigvee_{\text{some } d} \Sigma^d BP$$

$$BP^* \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$$

Hazewinkel generators:

$$\ell_1 = \frac{v_1}{p}, \quad \ell_2 = \frac{v_2}{p} + \frac{v_1^{1+p}}{p^2},$$

$$\ell_3 = \frac{v_3}{p} + \frac{v_1 v_2^p + v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3}, \text{ etc.}$$

Araki generators:

$$\ell_1 = \frac{v_1}{p - p^p}, \text{ etc.}$$

$\log_{BP}$, $\exp_{BP}$, and formal sum

Rationally, every formal group law is isomorphic to the additive formal group

$$x +_F y = \log_F^{-1}(\log_F(x) + \log_F(y))$$

$$\begin{aligned} \log_{BP}(t) &= t + \ell_1 t^p + \ell_2 t^{p^2} + \cdots \quad \text{(}p\text{-typical)} \\ \exp_{BP}(t) &= \log_{BP}^{-1}(t) \end{aligned}$$

$$\begin{aligned} \xi +_{BP} x &= \exp_{BP}(\, \log_{BP}(\xi) + \log_{BP}(x) \,) = \xi + x + \cdots \\ [i]\xi &= i\xi + \cdots \\ &= \xi \cdot \langle i \rangle \xi \end{aligned}$$

H_∞ ring structure for BP?

Consider $P_{C_p} : MU^{2i}(pt.) \to MU^{2pi}(BC_p)$

$$MU^*(BC_p) \cong MU^*[\![\xi]\!]/[p]\xi \xrightarrow{q_*} MU^*[\![\xi]\!]/\langle p\rangle\xi$$
$$BP^*(BC_p) \cong BP^*[\![\xi]\!]/[p]\xi \xrightarrow{q_*} BP^*[\![\xi]\!]/\langle p\rangle\xi$$

$$\begin{array}{ccccc} MU^{2*} & \xrightarrow{P_{C_p}} & MU^{2p*}[\![\xi]\!]/[p]\xi & \xrightarrow{q_* a_0^{2n}} & MU^{2p*+4n(p-1)}[\![\xi]\!]/\langle p\rangle\xi \\ \downarrow r_* & & \downarrow r_* & & \downarrow r_* \\ BP^{2*} & \overset{P_{C_p}}{\dashrightarrow} & BP^{2p*}[\![\xi]\!]/[p]\xi & \xrightarrow{q_* a_0^{2n}} & BP^{2p*+4n(p-1)}[\![\xi]\!]/\langle p\rangle\xi \end{array}$$

Calculate $MC_n = r_* q_* a_0^{2n} P_{C_p}[\mathbb{C}P^n]$
for $n \neq p^k - 1$

H_∞ ring structure for BP?

If the map $r : MU \to BP$ carries an H_∞ structure, then the dotted arrow makes the diagram commute and $MC_n = 0$ for $n \neq p^k - 1$

$$\begin{array}{ccccc}
MU^{2*} & \xrightarrow{P_{C_p}} & MU^{2p*}[\![\xi]\!]/[p]\xi & \xrightarrow{q_* a_0^{2n}} & MU^{2p*+4n(p-1)}[\![\xi]\!]/\langle p\rangle\xi \\
\downarrow r_* & & \downarrow r_* & & \downarrow r_* \\
BP^{2*} & \overset{P_{C_p}}{\dashrightarrow} & BP^{2p*}[\![\xi]\!]/[p]\xi & \xrightarrow{q_* a_0^{2n}} & BP^{2p*+4n(p-1)}[\![\xi]\!]/\langle p\rangle\xi
\end{array}$$

$$MC_n = r_* q_* a_0^{2n} P_{C_p}[\mathbb{C}P^n]$$

The same argument applies to any map of ring spectra $MU \xrightarrow{f} E \to BP$.

P_{C_p}: Quillen's cyclic product formula

$$MU^*(\mathbb{C}P^\infty) \cong MU^*[\![x]\!]$$
$$MU^*(BC_p \times \mathbb{C}P^\infty) \cong MU^*[\![x,\xi]\!]/[p]\xi$$

$$BP^*(\mathbb{C}P^\infty) \cong BP^*[\![x]\!]$$
$$BP^*(BC_p \times \mathbb{C}P^\infty) \cong BP^*[\![x,\xi]\!]/[p]\xi$$

$$r_* P_{C_p}(x) = \prod_{i=0}^{p-1} \left([i]\xi +_{BP} x\right) = \sum_{i\geq 0} a_i(\xi) x^{i+1}$$

This defines the classes a_i
a_0 is the MU-Euler class of the reduced regular representation of C_p

P_{C_p} on $[\mathbb{C}P^n]$

Theorem (Quillen)

$$a_0^{2n} P_{C_p}[\mathbb{C}P^n] = \sum_{|\alpha|=n} a^\alpha s_\alpha[\mathbb{C}P^n]$$

multi-indices $\alpha = (\alpha_0, \alpha_1, \ldots)$

$$a^\alpha = a_0^{\alpha_0} a_1^{\alpha_1} \cdots$$

$$|\alpha| = \sum_{i \geq 0} \alpha_i \qquad |\alpha|' = \sum_{i \geq 1} i\alpha_i$$

Adams: $s_\alpha[\mathbb{C}P^n] = coeff_{n,\alpha}[\mathbb{C}P^{n-|\alpha|'}]$ (modified multinomial coefficient)

P_{C_p}

Putting these together gives an explicit formula for $MC_n = r_* q_* a_0^{2n} P_{C_p}[\mathbb{C}P^n]$ (McClure):

$$MC_n = a_0^{2n+1} \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \cdot \left(\sum_{i \geq 0} a_i z^i\right)^{-(n+1)} [z^k]$$

$g(z)[z^k]$ extracts
the coefficient of z^k in $g(z)$

P_{C_p} $p = 2$

$$a_0(\xi) = \xi$$
$$a_1(\xi) = 1 - v_1\xi + v_1^2\xi^2 + (-2v_1^3 - 2v_2)\xi^3 + (3v_1^4 + 4v_1v_2)\xi^4 + (-4v_1^5 - 6$$
$$a_2(\xi) = v_1^2\xi + (-4v_1^3 - 3v_2)\xi^2 + (10v_1^4 + 11v_1v_2)\xi^3 + (-21v_1^5 - 28v_1^2v_2)\xi$$
$$a_3(\xi) = (-2v_1^3 - 2v_2)\xi + (10v_1^4 + 11v_1v_2)\xi^2 + (-34v_1^5 - 43v_1^2v_2)\xi^3 + (10$$
$$a_4(\xi) = (3v_1^4 + 4v_1v_2)\xi + (-21v_1^5 - 28v_1^2v_2)\xi^2 + (101v_1^6 + 164v_1^3v_2 + 34$$
$$a_5(\xi) = (-4v_1^5 - 6v_1^2v_2)\xi + (43v_1^6 + 75v_1^3v_2 + 18v_2^2)\xi^2 + (-275v_1^7 - 551$$
$$a_6(\xi) = (6v_1^6 + 12v_1^3v_2 + 4v_2^2)\xi + (-88v_1^7 - 190v_1^4v_2 - 89v_1v_2^2 - 14v_3)\xi^2$$
$$a_7(\xi) = (-10v_1^7 - 24v_1^4v_2 - 14v_1v_2^2 - 4v_3)\xi + (169v_1^8 + 420v_1^5v_2 + 257v_1^2$$
$$a_8(\xi) = (15v_1^8 + 40v_1^5v_2 + 28v_1^2v_2^2 + 8v_1v_3)\xi + (-312v_1^9 - 880v_1^6v_2 - 688$$

P_{C_p} $\quad p = 3$

$$a_0(\xi) = 2\xi^2 - 2v_1\xi^4 + 8v_1^2\xi^6 - 40v_1^3\xi^8 + (170v_1^4 - 170v_2)\xi^{10} + (-648v_1^5$$

$$a_1(\xi) = 3\xi - 8v_1\xi^3 + 36v_1^2\xi^5 - 216v_1^3\xi^7 + (1148v_1^4 - 944v_2)\xi^9 + (-5352$$

$$a_2(\xi) = 1 - 9v_1\xi^2 + 63v_1^2\xi^4 - 491v_1^3\xi^6 + (3336v_1^4 - 2331v_2)\xi^8 + (-1929$$

$$a_3(\xi) = -3v_1\xi + 53v_1^2\xi^3 - 606v_1^3\xi^5 + (5466v_1^4 - 3396v_2)\xi^7 + (-40124v_1^5$$

$$a_4(\xi) = 21v_1^2\xi^2 - 435v_1^3\xi^4 + (5547v_1^4 - 3248v_2)\xi^6 + (-53343v_1^5 + 1099$$

$$a_5(\xi) = 3v_1^2\xi - 179v_1^3\xi^3 + (3588v_1^4 - 2142v_2)\xi^5 + (-47382v_1^5 + 94662v_1$$

$$a_6(\xi) = -38v_1^3\xi^2 + (1454v_1^4 - 994v_2)\xi^4 + (-28406v_1^5 + 58352v_1v_2)\xi^6 +$$

$$a_7(\xi) = -3v_1^3\xi + (341v_1^4 - 324v_2)\xi^3 + (-11256v_1^5 + 25956v_1v_2)\xi^5 + (17$$

$$a_8(\xi) = (36v_1^4 - 72v_2)\xi^2 + (-2748v_1^5 + 8268v_1v_2)\xi^4 + (67120v_1^6 - 5189$$

P_{C_p} $p = 5$

$$a_0(\xi) = 24\xi^4 - 1680 v_1 \xi^8 + 370008 v_1^2 \xi^{12} - 123486336 v_1^3 \xi^{16} + 49940181$$
$$a_1(\xi) = 50\xi^3 - 5430 v_1 \xi^7 + 1551072 v_1^2 \xi^{11} - 636927168 v_1^3 \xi^{15} + 3065334$$
$$a_2(\xi) = 35\xi^2 - 7328 v_1 \xi^6 + 2893808 v_1^2 \xi^{10} - 1508394320 v_1^3 \xi^{14} + 880153$$
$$a_3(\xi) = 10\xi - 5498 v_1 \xi^5 + 3207450 v_1^2 \xi^9 - 2188580410 v_1^3 \xi^{13} + 1576841$$
$$a_4(\xi) = 1 - 2550 v_1 \xi^4 + 2370055 v_1^2 \xi^8 - 2186482212 v_1^3 \xi^{12} + 198178597$$
$$a_5(\xi) = -750 v_1 \xi^3 + 1237150 v_1^2 \xi^7 - 1600089600 v_1^3 \xi^{11} + 186105245632$$
$$a_6(\xi) = -130 v_1 \xi^2 + 469174 v_1^2 \xi^6 - 889462830 v_1^3 \xi^{10} + 1357095174226 v$$
$$a_7(\xi) = -10 v_1 \xi + 129998 v_1^2 \xi^5 - 383662650 v_1^3 \xi^9 + 787791379990 v_1^4 \xi^{13}$$
$$a_8(\xi) = 25850 v_1^2 \xi^4 - 129787730 v_1^3 \xi^8 + 369983450960 v_1^4 \xi^{12} - 7862998$$

$\langle p \rangle \xi$

$$\langle 2 \rangle = 2 - \xi v_1 + 2\xi^2 v_1^2 + \xi^3 \left(-8 v_1^3 - 7 v_2\right) + \xi^4 \left(26 v_1^4 + 30 v_1 v_2\right) + \xi^5 \left(-8\right.$$

$$\langle 3 \rangle = 3 - 8\xi^2 v_1 + 72\xi^4 v_1^2 - 840\xi^6 v_1^3 + \xi^8 \left(9000 v_1^4 - 6560 v_2\right) + \xi^{10} \left(-88\right.$$

$$\langle 5 \rangle = 5 - 624\xi^4 v_1 + 390000\xi^8 v_1^2 - 341094000\xi^{12} v_1^3 + 347012281200\xi^{16}$$

P_{C_p} mod $\langle p \rangle$ $\quad p = 2$

$a_0(\xi) \equiv \xi$

$a_1(\xi) \equiv 1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + (v_1^6 + v_1^3 v_2 + v_2^2)\xi^6 + v_1^4 v_2\xi^7 + (v_1^8 + v_1^2 v$

$a_2(\xi) \equiv v_1^2\xi + v_2\xi^2 + v_1 v_2\xi^3 + v_1^5\xi^4 + v_1^3 v_2\xi^5 + (v_1^4 v_2 + v_1 v_2^2)\xi^6 + (v_1^8 + v$

$a_3(\xi) \equiv v_1^4\xi^2 + (v_1^6 + v_1^3 v_2 + v_2^2)\xi^4 + v_1^4 v_2\xi^5 + (v_1^8 + v_1^5 v_2)\xi^6 + (v_1^9 + v_1^6 v_2$

$a_4(\xi) \equiv v_1^4\xi + (v_1^6 + v_1^3 v_2)\xi^3 + (v_1^4 v_2 + v_1 v_2^2 + v_3)\xi^4 + (v_1^5 v_2 + v_1 v_3)\xi^5 + ($

$a_5(\xi) \equiv v_1^6\xi^2 + (v_1^7 + v_1^4 v_2 + v_1 v_2^2)\xi^3 + (v_1^8 + v_1^2 v_2^2 + v_1 v_3)\xi^4 + (v_2^3 + v_1^2 v_3$

$a_6(\xi) \equiv (v_1^7 + v_1 v_2^2)\xi^2 + (v_1^8 + v_1^2 v_2^2 + v_1 v_3)\xi^3 + v_2^3\xi^4 + (v_1^{10} + v_2 v_3)\xi^5 + ($

$a_7(\xi) \equiv v_1^9\xi^3 + (v_1^{11} + v_1^8 v_2 + v_1^5 v_2^2)\xi^5 + (v_1^3 v_2^3 + v_1^5 v_3)\xi^6 + (v_1^{13} + v_1^{10} v_2 +$

$a_8(\xi) \equiv v_1^8\xi + v_1^9\xi^2 + (v_1^{10} + v_1^4 v_2^2)\xi^3 + (v_1^{11} + v_1^2 v_2^3 + v_1^4 v_3)\xi^4 + (v_1^6 v_2^2 + v$

P_{C_p} mod $\langle p \rangle$ $\quad p = 3$

$$a_0(\xi) \equiv 2\xi^2 + v_1\xi^4 + 2v_1^3\xi^8 + \left(v_1^4 + v_2\right)\xi^{10} + 2v_1^5\xi^{12} + v_1^2v_2\xi^{14} + \left(v_1^7 + v\right.$$

$$a_1(\xi) \equiv 0$$

$$a_2(\xi) \equiv 1 \ \text{(gap)} + v_1^7\xi^{14} + v_1^4v_2\xi^{16} + \left(v_1^5v_2 + v_1v_2^2\right)\xi^{18} + 2v_1^6v_2\xi^{20} + \left(2v\right.$$

$$a_3(\xi) \equiv 0$$

$$a_4(\xi) \equiv 2v_1^3\xi^4 + \left(v_1^4 + v_2\right)\xi^6 + 2v_1^5\xi^8 + v_1^2v_2\xi^{10} + \left(v_1^7 + v_1^3v_2\right)\xi^{12} + \left(2v\right.$$

$$a_5(\xi) \equiv 0$$

$$a_6(\xi) \equiv v_1^3\xi^2 + 2v_2\xi^4 + \left(v_1^5 + v_1v_2\right)\xi^6 + \left(v_1^6 + 2v_1^2v_2\right)\xi^8 + \left(2v_1^7 + v_1^3v_2\right)$$

$$a_7(\xi) \equiv 0$$

$$a_8(\xi) \equiv v_1^9\xi^{12} + 2v_1^{10}\xi^{14} + \left(2v_1^{11} + v_1^7v_2\right)\xi^{16} + \left(v_1^{12} + 2v_1^8v_2 + 2v_1^4v_2^2 + 2\right.$$

P_{C_p} mod $\langle p \rangle$ $p = 5$

$$a_0(\xi) \equiv 4\xi^4 + v_1\xi^8 + 4v_1^5\xi^{24} + (v_1^6 + v_2)\xi^{28} + 4v_1^9\xi^{40} + v_1^4 v_2\xi^{44} + v_1^5 v_2\xi^{48}$$
$$a_1(\xi) \equiv 0$$
$$a_2(\xi) \equiv 0$$
$$a_3(\xi) \equiv 0$$
$$a_4(\xi) \equiv 1 \ \text{(gap)} + 4v_1^{12}\xi^{48} + (v_1^{13} + 2v_1^7 v_2)\xi^{52} + 4v_1^2 v_2^2\xi^{56} + v_1^3 v_2^2\xi^{60} + 2v$$
$$a_5(\xi) \equiv 0$$
$$a_6(\xi) \equiv 0$$
$$a_7(\xi) \equiv 0$$
$$a_8(\xi) \equiv 4v_1^5\xi^{16} + (v_1^6 + v_2)\xi^{20} + 4v_1^9\xi^{32} + v_1^4 v_2\xi^{36} + v_1^5 v_2\xi^{40} + 2v_1^{13}\xi^{48} +$$

Classes a_i are zero unless i is divisible by $p-1$.

- $C_p^\times$ acts on BC_p
- In $BP^*(BC_p)$ an element $w \in C_p^\times$ acts on $[i]\xi$ by $[i]\xi \mapsto [wi]\xi$
- The cyclic product $\prod_{i=1}^{p-1}([i]\xi +_{BP} x)$ is $C_p^\times$-invariant
- $a_i \in BP^{2(p-i-1)}(BC_p)^{C_p^\times}$
- $H^*(BC_p)^{C_p^\times} \cong \mathbb{Z}/p[\xi^{(p-1)}]$ is concentrated in degrees divisible by $2(p-1)$
- Atiyah-Hirzebruch $\Rightarrow$ non-zero a_i are concentrated in degrees divisible by $2(p-1)$
- $\Rightarrow (p-1)|i$

Sparseness for MC_n

The obstructions MC_n are non-zero only if n is divisible by $p-1$.

- $p-1$ is one less than a power of p
- $2(p-1)$ is not of the form p^k-1
  $r_*[\mathbb{C}P^{2(p-1)}] = 0$ in BP^*

First case of interest: $n = 2(p-1)$

$$\begin{aligned} MC_{2(p-1)}(\xi) &= a_0^{2p-4} r_*[\mathbb{C}P^{(p-1)}]\left(-(2p-1)a_0 a_{(p-1)}\right) \\ &\quad + a_0^{2p-4} r_*[\mathbb{C}P^0]\left(-(2p-1)a_0 a_{2(p-1)} + p(2p-1)a_{(p-1)}^2\right) \\ &= (2p-1)a_0^{2p-4}\left(-v_1 a_0 a_{(p-1)} - a_0 a_{2(p-1)} + p a_{(p-1)}^2\right) \end{aligned}$$

$[\mathbb{C}P^0] = 1$ and $r_*[\mathbb{C}P^{p-1}] = v_1$

The obstructions MC_n $p = 2$

$$MC_1(\xi) \equiv \xi^2 v_1^2 + \xi^3 v_2 + \xi^4 \left(v_1^4 + v_1 v_2\right) + \xi^7 \left(v_1^7 + v_3\right) + \xi^8 \left(v_1^8 + v_1 v_3\right) + \xi^9 \left(\right.$$

$$MC_2(\xi) \equiv \xi^6 \left(v_1^6 + v_2^2\right) + \xi^7 \left(v_1^7 + v_3\right) + \xi^8 \left(v_1^5 v_2 + v_1 v_3\right) + \xi^9 v_2^3 + \xi^{10} \left(v_1^4 v_2^2\right.$$

$$MC_3(\xi) \equiv \xi^6 v_1^6 + \xi^7 \left(v_1^4 v_2 + v_1 v_2^2\right) + \xi^8 \left(v_1^8 + v_1^5 v_2 + v_1 v_3\right) + \xi^{10} \left(v_1^{10} + v_1^7\right.$$

$$MC_4(\xi) \equiv \xi^{10} v_1^4 v_2^2 + \xi^{11} \left(v_1^{11} + v_1^8 v_2 + v_1^5 v_2^2 + v_1^4 v_3\right) + \xi^{12} \left(v_1^9 v_2 + v_1^3 v_2^3 + \right.$$

The obstructions MC_n $2 < p \leq 13$

$p = 3 : MC_4(\xi) \equiv 2v_1^9\xi^{22} + 2v_1^{10}\xi^{24} + 2v_1^7 v_2\xi^{26} + (2v_1^8 v_2 + v_1^4 v_2^2)\xi^{28} + O$

$p = 5 : MC_8(\xi) \equiv 3v_1^{16}\xi^{88} + (4v_1^{17} + v_1^{11} v_2)\xi^{92} + (3v_1^{18} + 4v_1^6 v_2^2)\xi^{96} + O$

$p = 7 : MC_{12}(\xi) \equiv 4v_1^{22}\xi^{192} + (4v_1^{23} + 2v_1^{15} v_2)\xi^{198} + (6v_1^{24} + 4v_1^{16} v_2 + 5$

$p = 11 : MC_{20}(\xi) \equiv 9v_1^{34}\xi^{520} + (8v_1^{35} + 6v_1^{23} v_2)\xi^{530} + (7v_1^{36} + v_1^{24} v_2 + 5v$

$p = 13 : MC_{24}(\xi) \equiv 11v_1^{40}\xi^{744} + (6v_1^{41} + 6v_1^{27} v_2)\xi^{756} + O(\xi^{768})$

The obstructions MC_n $p > 13$

Conjecture (strong form)

For any prime p, the coefficients of

$$v_1^{3p+1}\xi^{5p^2-8p+3} \quad \text{and} \quad v_1^{2p+1}v_2\xi^{5P^2-7p+2}$$

are non-zero in $MC_{2(p-1)}$.

The obstructions MC_n

p	$\deg(MC_{2(p-1)})$	$5p^2 - 8p + 3$	term	time
2	0	7	$v_1^7 \xi^7$	fast
3	8	24	$2 v_1^{10} \xi^{24}$	fast
5	48	88	$3 v_1^{16} \xi^{88}$	fast
7	120	192	$4 v_1^{22} \xi^{192}$	$\sim$ 1/2 day
11	360	520	$9 v_1^{34} \xi^{520}$	$\sim$ 4 days
13	528	744	$11 v_1^{40} \xi^{744}$	$\sim$ 22 days
17	960	1312	**??**	**??**

Conclusion

Theorem (J. – Noel)

Suppose $f : MU_{(p)} \to E$ is map of H_∞ ring spectra satisfying:

1. f factors through Quillen's map to BP.
2. f induces a Landweber exact MU_*-module structure on E_*.
3. **Small Prime Condition:** $p \in \{2, 3, 5, 7, 11, 13\}$.

then $\pi_* E$ is a $\mathbb{Q}$-algebra.

Application: The standard complex orientations on E_n, $E(n)$, $BP\langle n\rangle$, and BP do not respect power operations;
The corresponding MU-ring structures do not rigidify to commutative MU-algebra structures.

Thank You!