NOTES ON HOMOTOPIC DESCENT

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Abstract. These are notes from our reading group on homotopic descent, mainly following the preprint of Kathryn Hess [Hes10]. The talks were given by Michael Ching, while the notes were typed by Niles Johnson and Tiago Macedo.

source file: `UGA-descent-notes.tex`

1. Introduction

We consider the following (perhaps overly general) motivating question(s): Let F be a functor $\mathscr{C} \to \mathscr{D}$, and suppose y is an object in $\mathscr{D}$ (or f is a morphism in $\mathscr{D}$).

- Under what conditions is y (or f) in the essential image of F?
- What additional data is needed to characterize the essential image?
- Equivalently, let h_y be a functor $\mathscr{C} \to \mathcal{S}et$; under what conditions is h_y representable?

Below we give some of our motivating examples, using these to refine the descent question.

1.1. **Topology.** Let $f : Z \to Y$ be a map of topological spaces, and let $\mathscr{C}_Z, \mathscr{C}_Y$ denote the categories of spaces over Z and Y, respectively. Then the pull-back along f induces a functor $f^* : \mathscr{C}_Y \to \mathscr{C}_Z$.

The descent question is: Given $E \xrightarrow{p} Z$, when is there some $D \to Y$ such that $E \cong f^*(D) = Z \times_Y D$?

The following special case is already of considerable interest: Suppose that $\{U_\alpha\}$ is an open cover of Y, and let $Z = \coprod_\alpha U_\alpha$ with f the natural quotient map. Then the descent question concerns whether a given family of local spaces $E_\alpha \xrightarrow{p_\alpha} U_\alpha$ is obtained as the restriction of some global space E over Y. In this case E is uniquely determined if it exists, and we rephrase the question as being whether the E_α "assemble" to a global space over Y.

The theorem is that this happens precisely when we are given isomorphisms

$$f_{\alpha\beta} \colon E_\alpha|_{U_\alpha \cap U_\beta} \to E_\beta|_{U_\alpha \cap U_\beta}$$

with agreement on triple intersections. We will look to see this answer arise from the general theory we develop.

1.2. **Algebra.** Let $\phi\colon B \to A$ be a map of rings, and let $\phi_* = - \otimes_B A\colon \mathcal{M}od_B \to \mathcal{M}od_A$ be the extension of scalars functor.

The descent question is: Given an A-module N, when is there some B-module M such that $N \cong M \otimes_B A$?

Here too we have a special case of considerable interest: Let $\phi\colon k \to K$ be a Galois field extension, and let V be a K-vector space. Then the descent question is whether $V \cong U \otimes_k K$ for some k-vector space U.

The theorem is that this happens precisely when V has an action of the Galois group G of K/k, such that for each $\sigma \in G$

$$\sigma(\lambda v) = \sigma(\lambda)\sigma(v), \qquad \lambda \in K,\ v \in V.$$

In this case, $V \cong V^G \otimes_k K$.

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Moreover, this gives an equivalence of categories between those K-vector spaces in the essential image of ϕ_* and those with such a G-action – this is the kind of answer we will look for in general.

Note. The functors f^* and ϕ_* are both adjoints, but on different sides; the theory we develop will have a duality corresponding to left versus right adjoints.

1.3. **Fibered Categories.**

Definition 1.4. A fibered category consists of a functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ such that, for each morphism $f\colon x \to y$ in $\mathbb{D}$ there is a naturally induced functor

$$f^* : \mathbb{C}_y \to \mathbb{C}_x$$

where $\mathbb{C}_x = \mathbb{F}^{-1}(x)$.

Both of the examples above arise in this way, and we will look to answer the descent question "within $\mathbb{C}$" in each case.

2. Co/Monadic Descent

Let $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ be a pair of functors and suppose F is left-adjoint to G. Then the composite functor $K = FG$ is a comonad on $\mathscr{D}$, and the composite $T = FG$ is a monad on $\mathscr{C}$. In this section we study the descent question for F in terms on the comonad K; note however, that if F is right-adjoint to a functor L, then we have a monad $T = FL$ and a corresponding theory of monadic descent.

2.1. **Comonadic Structure Maps.**

Recall that the structure maps for K arise from the unit and counit of the adjunction $F \dashv G$:

$$\left.\begin{array}{l} \eta\colon id_{\mathscr{C}} \to GF \\ \varepsilon\colon FG \to id_{\mathscr{D}} \\ \quad \text{with compatibility} \end{array}\right\} \rightsquigarrow \left\{\begin{array}{l} \mu\colon K \to KK = F(GF)G \\ \varepsilon\colon K \to id_{\mathscr{D}} \\ \quad \text{with coassociativity} \\ \quad \text{and counitality} \end{array}\right.$$

Definition 2.2. An object y in $\mathscr{D}$ is called a coalgebra over a comonad K if there is a coaction map

$$y \to Ky$$

in $\mathscr{D}$ which is compatible in the natural way with the comultiplication μ and the counit ε. The category of K-coalgebras in $\mathscr{D}$ is denoted $\mathscr{D}_K$.

Recall that the functor F naturally "lifts" to the subcategory of K-coalgebras: for any x in $\mathscr{C}$, $F(x)$ has an inherent K-coaction. This lift is called the canonical functor

$$\begin{tikzcd} & \mathscr{D}_K \arrow[d] \\ \mathscr{C} \arrow[ur, "Can_K"] \arrow[r, "F"'] & \mathscr{D} \end{tikzcd}$$

and we can see a natural rephrasing of the descent question in this case: When is an object or morphism y in the subcategory of K-coalgebras?

A relevant further definition is that of "comonadic" functors.

Definition 2.3 (Comonadic)**.** A functor F as above is called comonadic if Can_K is an equivalence of categories.

Note. If $\mathscr{C}$ has limits, then the canonical functor Can_K always has a right adjoint, $Prim_K$, defined as the equalizer of the K-coaction and the unit of the adjunction:

$$Prim_K(y) = \text{equalizer}\,(\ G(y) \rightrightarrows GFG(y)\ )$$

Theorem 2.4 (Beck)**.** *The functor Can_K is fully faithful if and only if for each object $x \in \mathscr{C}$ the composite $Prim_K\, Can_K(x)$ is isomorphic to x. That is, the natural map from x to the equalizer is an isomorphism:*

$$x \xrightarrow{\cong} eq(\ GF(x) \rightrightarrows GFGF(x)\ ).$$

2.5. **Algebra.** We continue the example from Section 1.2 using comonadic language. As above, we let $\phi\colon B \to A$ be a map of rings, and let $F = \phi_*$ be the extension of scalars functor. Then F has a right adjoint G given by restriction of scalars, and we let $K = FG\colon \mathcal{Mod}_A \to \mathcal{Mod}_A$ be the induced comonad. Note that for an A-module N, $K(N)$ can be identified as

$$K(N) = G(N) \otimes_B A = N \otimes_A (A \otimes_B A).$$

In fact, the comonadic structure on K induces an A-co-ring structure on $A \otimes_B A$. The comultiplication is

$$\begin{aligned} A \otimes_B A &\to (A \otimes_B A) \otimes_A (A \otimes_B A) = A \otimes_B A \otimes_B A \\ a \otimes a' &\mapsto a \otimes 1 \otimes a' \end{aligned}$$

and the counit is

$$\begin{aligned} A \otimes_B A &\to AA \\ a \otimes a' &\mapsto aa' \end{aligned}$$

Proposition 2.6. *Let N be an A-module. Then N is an algebra over the monad K if and only if N is a comodule over the co-ring $A \otimes_B A$.*

Thus we see that for N to be in the essential image of F, N must be a comodule over $A \otimes_B A$. We may also wish to ask whether, in this case, N lifts to a unique element of $\mathcal{Mod}_B$ or not; this is addressed by the monadicity of F. We have the following facts about comonadicity in this case:

Proposition 2.7 (Grothendieck, Mesablishvili, Joyal-Tierry, Olivier)**.**

i. If A is faithfully flat over B, then $- \otimes_B A$ is comonadic.

ii. The functor $- \otimes_B A$ is comonadic if and only if A is pure over B, that is, the induced map $M \otimes_B B \to M \otimes_B A$ is injective for all B-modules M.

Exercise 2.8. If $\phi\colon k \to K$ is a Galois extension of fields, what does it mean for a K-vector space to be a comodule over $K \otimes_k K$?

3. Comonadic Codescent for Bundles

Let $f : X \to Y$ be a continuous function between topological spaces. We can define a functor

$$F = f^*\colon \mathcal{Top}/Y \to \mathcal{Top}/X$$

by pulling back a space over Y, $E \xrightarrow{p} Y$, along f. Last time, the following question was raised: Suppose we have an open covering

$$Y = \bigcup_\alpha U_\alpha$$

and let $X = \coprod_\alpha U_\alpha$. The descent machinery from the previous section yeilds a description in terms of glueing of bundles over Y, and we present excercises to walk the reader through this description.

Consider the left adjoint

$$H : \mathcal{Top}/X \to \mathcal{Top}/Y$$

of F. Then $T = FH$ is a monad in $\mathcal{T}op/X$,

$$T : \begin{pmatrix} E \\ \downarrow \\ X \end{pmatrix} \mapsto \begin{pmatrix} E \times_Y X \\ \downarrow \\ X \end{pmatrix}.$$

Note $E \times_Y X \cong E \times_X (X \times_Y X)$, as described by the following pullbacks:

$$\begin{tikzcd} E \times_Y X \arrow[r] \arrow[d] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ E \arrow[r] & X \arrow[r] & Y \end{tikzcd}$$

Exercise 3.1. Show that the image of F is contained in the T-algebras. That is, there is a lift, Can^T, as shown below:

$$\begin{tikzcd} & (\mathcal{T}op/X)^T \arrow[d] \\ \mathcal{T}op/Y \arrow[ur, "Can^T"] \arrow[r, "F"'] & \mathcal{T}op/X \end{tikzcd}$$

where

$$Can^T \begin{pmatrix} D \\ p\big\downarrow \\ Y \end{pmatrix} = D \times_Y X.$$

Exercise 3.2. $X \times_Y X$ is a monoid for $\times_Y$, with product given by

$$\begin{aligned} (X \times_Y X) \times_Y (X \times_Y X) &\to X \times_Y X \\ ((x_1, x_2), (x_3, x_4)) &\mapsto (x_1, x_4) \end{aligned}$$

Exercise 3.3. The indecomposables, Q, gives a functor from $(\mathcal{T}op/X)^T$ to $\mathcal{T}op/Y$:

$$Q \begin{pmatrix} E \\ \downarrow \\ X \end{pmatrix} = \operatorname{coeq}\Big(E \times_X (X \times_Y X) \rightrightarrows E \Big)$$

where the two maps in the coequalizer are the projection to E and the action of $(X \times_Y X)$.

Now F is monadic if Can^T is an equivalence of categories.

Exercise 3.4. Let $\{U_\alpha\}$ be an open cover of Y, and let $X = \coprod_\alpha U_\alpha$ with f given by the quotient map. Then the pullback functor F is monadic. The condition for $E \xrightarrow{p} X$ to be an algebra for the monad T is equivalent to the classical "agreement on intersections" condition, where $E_\alpha = p^{-1}(U_\alpha)$.

4. A Framework for Homotopic Descent

Now we attempt to motivate and understand the homotopic version of descent theory described by Hess [Hes10]. In this section we give a vague and intuitive motivation for the setting in which we will address the homotopic descent question.

Recall that, given a functor $F : \mathscr{C} \to \mathscr{D}$, we defined a category $\mathscr{E}$ which (is sufficiently interesting and) admits a forgetful functor $\mathscr{E} \to \mathscr{D}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathscr{E} \arrow[d, "Forget"] \\ \mathscr{C} \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & \mathscr{D}. \end{tikzcd}$$

Then the descent questions can be rephrased in terms of the functor $\tilde{F}$. Is $\tilde{F}$:

- Fully faithful?
- Essentially surjective?
- An equivalence of categories?

As described in Section 2, we can answer these questions when we have monadic or comonadic adjunctions, taking $\mathscr{E}$ to be the category $\mathscr{D}^T$ of T-algebras or $\mathscr{D}_K$ of K-coalgebras.

Now we turn to the question of homotopic descent. Suppose we have $\mathscr{C}, \mathscr{D}$ two categories with a *notion of homotopy* between maps. By *notion of homotopy* between maps, we mean an equivalence relation in the set of morphisms $\mathscr{C}(x, y)$, for all $x, y \in \mathscr{C}$, satisfying

- if f is equivalent to g, then hf is equivalent to hg, for all h,
- if f is equivalent to g, then fh is equivalent to gh, for all h.

If two morphisms $f, g \in \mathscr{C}(x, y)$ are equivalent, we denote

$$f \simeq g$$

and we then they are said to be *homotopic*.

Note. We are not imposing any extra restriction on this equivalence relation, nor any intuition. For the time being, we are interested in dealing simply with the abstract equivalence relation.

Examples 4.1.

- In the category $\mathcal{T}\!op$, we have the usual notion of homotopy.
- In the category of chain complexes over a ring R, we have the notion of chain homotopy.
- In a Quillen model category, we have weak equivalences.
- In an arbitrary category, we can define homotopy as equality.

Now let's try to rephrase the descent questions. One obvious way to do this is the following: Take a functor $F : \mathscr{C} \to \mathscr{D}$ between categories with a notion of homotopy which satisfies

$$f \simeq g \Rightarrow F(f) \simeq F(g), \quad \forall\, f, g \in \mathscr{C}(x, y),\ x, y \in \mathscr{C}.$$

Then ask: What additional structure on the category $\mathscr{D}$ we should impose so that, for any $y \in \mathscr{D}$, we can recover a unique (up to homotopy equivalence) $x \in \mathscr{C}$, such that $F(x) \simeq y$.

Now, let us try to answer it two different frameworks. First in a naïve way and then in a better one.

4.2. **One Answer.** If $\mathscr{C}$ is a category with a notion of homotopy, then we can define the homotopy category of $\mathscr{C}$ in the following way. Let $Ho(\mathscr{C})$ denote the category, whose objects are the same as the objects of $\mathscr{C}$ and the morphisms are equivalence classes of morphisms of $\mathscr{C}$. This means (roughly) that we quotient the set of morphisms by the homotopy equivalence relation, but preserve the objects.

Note. We are ignoring the set-theoretic problems with such a vague idea, since we will move toward a more precise answer soon.

Note that any functor $F : \mathscr{C} \to \mathscr{D}$ between categories with a homotopy notion which preserves the homotopy relation induces a functor $Ho(F) : Ho(\mathscr{C}) \to Ho(\mathscr{D})$. Now, one might try to carry out the descent theory of Section 2 with $Ho(F)$, *i.e.*, ask when is $Ho(F)$ (co)monadic. If there is a (usual) category $\mathscr{E}$, such that the following diagram

$$\begin{tikzcd} & \mathscr{E} \arrow[d, "Forget"] \\ Ho(\mathscr{C}) \arrow[ur, "\Phi"] \arrow[r, "Ho(F)"'] & Ho(\mathscr{D}) \end{tikzcd}$$

is commutative and Φ is fully faithful, essentially surjective or an equivalence of categories, then we (might[1]) have an answer for our *homotopic descent question*. However in many cases of interest, the induced functor $Ho(F)$ will fail to be monadic, and thus we look for more subtle answers. For example, classical descent for derived categories is well-known to fail. One indication of the difficulty can be seen in the observation that co/equalizers often fail to preserve homotopy.

4.3. **A Better Answer.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between categories with homotopy notion (which preserves the homotopy relation). Let us try to construct a commutative diagram

$$\begin{tikzcd} & \mathscr{E} \arrow[d, "Forget"] \\ \mathscr{C} \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & \mathscr{D}, \end{tikzcd}$$

where $\mathscr{E}$ is a category with the notion of homotopy and $\tilde{F}$ is an "*equivalence of categories with notion of homotopy*".

Lets see what happens in the (co)monadic point of view. If F has a left adjoint H, which preserves homotopies, then we could try to take $\mathscr{E} = \mathscr{D}^T$, where $T = FH$. Then the natural question is:

Question 4.4. Is there a notion of homotopy in $\mathscr{D}^T$?

Answer: Suppose T preserves homotopy relation, i.e.

$$f \simeq g \Rightarrow T(f) \simeq T(g).$$

Then, given any maps $f, g \in \mathscr{D}^T(x, y)$, when is $f \simeq g$?

A first guess is: $f \simeq g \in \mathscr{D}^T$ iff $f \simeq g \in \mathscr{D}$. But this will not be the right notion unless F "reflects homotopies", i.e. unless

$$f \simeq g \in \mathscr{C} \iff F(f) \simeq F(g) \in \mathscr{D}, \quad \forall f, g.$$

The problem here is analogous to the one we have in topological spaces with a distinguished point, namely that homotopies should be homotopies *through base-point-preserving maps*. In this case, we ask that we can find homotopies *through maps of T-algebras*. Of course it is not clear what that should mean if our homotopies are not paths in some mapping space, but we ignore that question for the moment.

Here is another way of tracking the extra structure $m \colon TA \to A$: Note that we have the following commutative diagrams for $f \simeq g$.

$$\begin{tikzcd} Tx \arrow[r, "Tf"] \arrow[d, "\theta_x"'] & Ty \arrow[d, "\theta_y"] \\ x \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} Tx \arrow[r, "Tg"] \arrow[d, "\theta_x"'] & Ty \arrow[d, "\theta_y"] \\ x \arrow[r, "g"'] & y \end{tikzcd}$$

[1]Ignoring, of course, the question of lifting from $Ho(\mathscr{C})$ to $\mathscr{C}$.

These yield "two different homotopies" $\theta_y Tf \simeq g\theta_x$:

$$f \simeq g \Rightarrow \theta_y Tf = f\theta_x \overset{H_1}{\sim} g\theta_x$$

and

$$Tf \simeq Tg \Rightarrow \theta_y Tf \overset{H_2}{\sim} \theta_y Tg = g\theta_y.$$

However, they should be the same thing, i.e. we want to say that $f \simeq g \in \mathscr{D}^T$ if these are the same homotopy, but we have to keep track of the homotopies in the original category $\mathscr{D}$. . . . $\square$

Remark 4.5. An analogy: the points 1 and -1 are in the same path component of $S^1 \subset \mathbb{C}$, but there are two *different* paths which exhibit this fact. Something like this is a situation we may want to avoid.

4.6. ***Top*-Categories.** To make sense of the idea presented above, we will build some extra data on $\mathscr{D}$ to allow for specific choices if homotopies. Following [Hes10] , we consider enrichment in *Top*.

Definition 4.7. A *Top-category* $\mathscr{C}$ is a category enriched over *Top*. In other words, it consists of

- a class of objects
- for any two objects x, y in $\mathscr{C}$, a topological space $\mathscr{C}(x, y)$,
- a continuous composition map
- an identity point $\mathrm{id}_x \in \mathscr{C}(x, x)$, for all objects x in $\mathscr{C}$,

satisfying the usual axioms of a category.

Examples 4.8.

- *Top* is a *Top*-category, where we endow $\mathscr{C}(x, y)$ with the compact-open topology.
- A Quillen model category is naturally enriched in simplicial sets (perhaps with some additional technical hypotheses), and the category of simplicial sets is equivalent to *Top*.
- An arbitrary category $\mathscr{C}$ with the set $\mathscr{C}(x, y)$ having the discrete or indiscrete topology is a *Top*-category.

The main idea is that a *Top*-category has a notion of homotopy between morphisms:

Definition 4.9. Let $\mathscr{C}$ be a *Top*-category. Two morphisms $f, g \in \mathscr{C}(x, y)$ are said to be *homotopic* iff f and g belong to the same path-component of $\mathscr{C}(x, y)$.

Examples 4.10.

- In the *Top*-category *Top* this corresponds to the usual notion of homotopy between continuous maps.
- In a Quillen model category (again, perhaps with technical hypotheses), the notion of homotopy coming from enrichment in simplicial sets agrees with the notion of homotopy arising from cylinder and path objects.

Note that the topological enrichment gives us a natural way of tracking "higher homotopies", since a homotopy between f and g is a path in the topological space $\mathscr{C}(x, y)$ between f and g. Therefore, we have a notion of whether two homotopies are homotopic or not.

Definition 4.11. Let $\mathscr{C}$ and $\mathscr{D}$ be *Top*-categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be *continuous* if

$$F : \mathscr{C}(x, y) \to \mathscr{D}(Fx, Fy)$$

is a continuous map, for all $x, y \in \mathscr{C}$.

Proposition 4.12 ([Hes10]). *Let $\mathscr{D}$ be a $\mathcal{T}op$-category and $T : \mathscr{D} \to \mathscr{D}$ a continuous monad. Then the category $\mathscr{D}^T$ of T-coalgebras is a $\mathcal{T}op$-category with*

$$\mathscr{D}^T(x, y) := \mathrm{eq}\left(\mathscr{D}(x, y) \underset{\theta_x}{\overset{(\theta_y \circ Tf)}{\rightrightarrows}} \mathscr{D}(Tx, y) \right), \quad \forall\ x, y \in \mathscr{D}^T.$$

Remark 4.13. The definition of $\mathscr{D}^T(x, y)$ in the above proposition incorporates the commutativity of

$$\begin{tikzcd} Tx \arrow[r, "Tf"] \arrow[d, "\theta_x"'] & Ty \arrow[d, "\theta_y"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

for $f \in \mathscr{D}(x, y)$ and formalizes the idea of homotopy through T-algebras.

4.14. **Monadic Homotopic Descent.** Let $F \dashv G$ be an enriched adjunction of continuous functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

with F left adjoint to G. Then the comonad $K = FG$ is continuous and we have

$$\begin{tikzcd} & \mathscr{D}_K \arrow[d, "Forget"] \\ \mathscr{C} \arrow[ur, "Can_K"] \arrow[r, "F"] & \mathscr{D} \end{tikzcd}$$

In this situation we have the (poorly-behaved) functor $Prim_k$, together with a (better-behaved) functor $Tot(\Omega^\bullet -)$ (see Section 6), obtained from a cobar contstruction and totalization of the resulting cosimplicial object. Our study of homotopic descent will aim to explain in what ways $Tot(\Omega^\bullet -)$ is better-behaved, and how to use this functor to address descent questions. We close this section with the homotopic version of Beck's condition.

Proposition 4.15. *If $\mathscr{C}$ has the necessary equalizers, Can_K has a right adjoint*

$$Prim_K \colon \mathscr{D}_K \to \mathscr{C}$$

given by

$$Prim_K(y) = eq(Gy \rightrightarrows y)$$

Proposition 4.16 (Beck Condition). *Suppose the natural map*

$$x' \xrightarrow{\eta_{x'}} eq(GFx' \rightrightarrows Fx')$$

is a homotopy equivalence in $\mathscr{C}$. Then the induced map

$$\mathscr{C}(x, x') \to \mathscr{D}_K(Fx, Fx')$$

is a homotopy equivalence for all $x \in \mathscr{C}$.

Proof. We have

$$\mathscr{C}(x, x') \xrightarrow{\simeq} \mathscr{C}(x, Prim_K Can_K(x')) \cong \mathscr{C}(Can_K x, Can_k x') = \mathscr{C}(Fx, Fx')$$

since $Can_K(x) = F(x)$. □

Remark 4.17. Beck's condition is (perhaps under some technical hypotheses) *necessary and sufficient* for F to induce equivalences on mapping spaces. Since this condition is not satisfied in interesting examples, we look for conditions which are weaker, and therefore we must not expect F to induce equivalences on mapping spaces in general. Of course it will be of interest to determine conditions on x' relative to F which do then guarantee such equivalences.

4.18. **Motivation, Revised.** [`This section not yet written ...`]

5. Cosimplicial Objects

`This section will contain a review of cosimplicial objects ...`

5.1. **The Cobar Complex.**

5.2. **Tot.**

6. The Descent Spectral Sequence

For a monad T on $\mathscr{C}$, we let $T^{\bullet+1}x$ denote the cosimplicial object

$$p \mapsto T^{p+1}x$$

with coface and codegeneracy maps given by the structure maps of T. When $T = GF$ is the monad arising from an adjunction (as in our case of interest)

$$\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$$

then there is an auxilary cosimplicial (cobar) construction on $\mathscr{D}$: $\Omega^\bullet y$ is the cosimplical object

$$p \mapsto [G(FG)^p](y)$$

with coface and codegeneracy maps given by the structure maps of the adjunction. Note that for $x \in \mathscr{C}$, $T^{\bullet+1}x = \Omega^\bullet Fx$.

Definition 6.1 (T-complete)**.** An object $x \in \mathscr{C}$ is *T-complete* if the natural unit map

$$\eta\colon x \to Tot(T^{\bullet+1}x)$$

is an equivalence in $\mathscr{C}$. The object $Tot(T^{\bullet+1}x)$ is the *T-completion* of x, denoted $x\widehat{_T}$.

Let $c^\bullet x$ denote the constant cosimplicial object:

$$p \mapsto x$$

with all coface and codegeneracy maps being the identity.

Definition 6.2 (strictly T-complete)**.** An object $x \in \mathscr{C}$ is *strictly T-complete* if the natural map

$$\eta^\bullet\colon c^\bullet x \to T^{\bullet+1}x$$

is an external cosimplicial strong deformation retract. That is, there is a cosimplicial section

$$\rho^\bullet\colon T^{\bullet+1}x \to c^\bullet x$$

such that, at each cosimplicial level,

$$\mathrm{id}_x = \rho^\bullet \circ \eta^\bullet\colon c^p x \to T^{p+1} \to c^p x$$

and $\eta \circ \rho$ is cosimplicially homotopic to $\mathrm{id}_{T^{\bullet+1}x}$.

$$\begin{tikzcd}
x \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] & x \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift left] & x \arrow[r, shift left=2] \arrow[r, dotted] \arrow[r, shift right=2] \arrow[d, shift left] & \cdots \\
Tx \arrow[r, shift left] \arrow[r, shift right] \arrow[u, dashed, shift left, "\rho"] & T^2x \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[u, dashed, shift left, "\rho"] & T^3x \arrow[r, shift left=2] \arrow[r, dotted] \arrow[r, shift right=2] \arrow[u, dashed, shift left, "\rho"] & \cdots
\end{tikzcd}$$

Remark 6.3. When x is strictly T-complete, the section $\rho^\bullet$ is something like a "homotopy T-algebra structure" on x: we have maps $\rho^p \colon T^{p+1}x \to x$ which are compatible with the cosimplicial structure maps on $T^{\bullet+1}x$, *i.e.*, the unit and product maps of T.

Proposition 6.4 (Homotopic Descent Criterion II)**.** *If x' is strictly T-complete ($T = GF$), then F induces an equivalence of mapping spaces*

$$\mathscr{C}(x, x') \xrightarrow{\simeq} \mathscr{D}_K(Fx, Fx')$$

Idea of proof.

$$\begin{tikzcd}
\mathscr{C}(x,x') = Tot\,\mathscr{C}(x, c^\bullet x') \arrow[r, shift right, "\eta^\bullet"'] \arrow[r, phantom, "\simeq (a)"] & Tot\,\mathscr{C}(x, T^{\bullet+1}x') \arrow[l, shift right, "\rho^\bullet"'] \arrow[d, "\cong"] \\
\mathscr{D}_K(Fx, Fx') = Tot\,\mathscr{D}_K(x, c^\bullet x') \arrow[r, shift right, "\eta^\bullet"'] \arrow[r, phantom, "\simeq (b)"] & Tot\,\mathscr{D}_K(Fx, FT^{\bullet+1}x') \arrow[l, shift right, "\rho^\bullet"']
\end{tikzcd}$$

The vertical isomorphism follows formally: Since each $T^{\bullet+1}x'$ is in the image of G, it suffices to prove, for any $y \in \mathscr{D}$,

$$\mathscr{C}(x, Gy) \xrightarrow{\cong} \mathscr{D}_K(Fx, FGy).$$

We see this using classical descent (the Beck criterion): $Gy \cong Prim_K Can_K Gy$ because the diagram below is a split equalizer, and therefore an equalizer:

$$\begin{tikzcd}
Gy \arrow[r] & GFGy \arrow[l, dashed, bend right, "\alpha"'] \arrow[r, shift left] \arrow[r, shift right] & GFGFGy \arrow[l, dashed, bend right, "\beta"']
\end{tikzcd}$$

where the dashed arrows are induced by the counit $FGy \to y$.

$\ldots \square$

Now that if x' is assumed only to be T-complete, the equivalence marked (a) still holds because right adjoints commute and hence $Tot\,\mathscr{C}(x, T^{\bullet+1}x') \cong \mathscr{C}(x, Tot\,T^{\bullet+1}x')$. Likewise, the vertical isomorphism induces $\mathscr{C}(x, Tot\,T^{\bullet+1}x') \cong \mathscr{D}_K(Fx, Tot\,FT^{\bullet+1}x')$. The equivalence marked (b), however, depends on having $\rho^\bullet$ defined at the cosimplicial level. The diagram below describes this situation:

$$\begin{tikzcd}
\mathscr{C}(x,x') \arrow[r, "\simeq"] \arrow[d] & \mathscr{D}_K(Fx, Tot\,FT^{\bullet+1}x') \\
\mathscr{D}_K(Fx, Fx') \arrow[r, "\simeq"] & \mathscr{D}_K(Fx, F(Tot\,T^{\bullet+1}x')) \arrow[u]
\end{tikzcd}$$

Thus we have a spectral sequence similar to the Bousfield-Kan spectral sequence comparing the two rows. For a refresher on the Bousfield-Kan spectral sequence, the notes of Bertrand Guillou [Gui07] are quite nice.

7. Applications

7.1. **The Adams Spectral Sequence.** Let $\mathscr{M}$ be a $\mathcal{T}op$-category which is also cotensored over $\mathcal{T}op$, and let $\phi\colon B \to A$ be a map of monoids in $\mathscr{M}$. Then we have the adjunction

$$\begin{tikzcd} \mathcal{M}od_B \arrow[r, shift left, "- \wedge_B A"] & \mathcal{M}od_A \arrow[l, shift left, "\phi^*"] \end{tikzcd}$$

and one can check that both $\mathcal{M}od_B$ and $\mathcal{M}od_A$ inherit the enrichment and cotensor over $\mathcal{T}op$ from $\mathscr{M}$. Moreover, the extension and restriction of scalars functors are continuous and form a continuous adjunction (*i.e.*, one enriched over $\mathcal{T}op$).

Now let $W = A \wedge_B A$ be the associated co-ring in $\mathcal{M}od_A$, and consider the descent question for:

$$\begin{tikzcd} & \mathcal{C}omod_W \arrow[d, "Forget"] \\ \mathcal{M}od_B \arrow[ur, "Can_W"] \arrow[r, shift left, "- \wedge_B A"] & \mathcal{M}od_A \arrow[l, shift left, "\phi^*"] \end{tikzcd}$$

Given $M, M' \in \mathcal{M}od_B$, we study the induced map of spaces

$$\mathcal{M}od_B(M, M') \to \mathcal{C}omod_W(M \wedge_B A, M' \wedge_B A).$$

We begin this study by writing the cobar construction for a W-comodule, N; this is a cosimplicial B-module:

$$\Omega^\bullet(N)\colon \phi^* N \rightrightarrows \phi^* N \wedge_B \phi^* A \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \phi^* N \wedge_B \phi^* A \wedge_B \phi^* A \substack{\longrightarrow\\\vdots\\\longrightarrow} \cdots$$

Note that $\phi^* N = N$ and $\phi^* A = A$. For each cosimplicial degree p, the first $p+1$ coface maps are maps of W-comodules, but the last one is merely a B-module map. Now taking $N = M \wedge_B A$ for some M, we have $\Omega^\bullet(M \wedge_B A) = T^{\bullet+1}(M)$, where $T = \phi^*(- \wedge_B A)$ is the associated monad. We proceed by considering the various completeness conditions in this case.

Example 7.2. Let $\mathscr{M}$ be the category of spectra, B the sphere spectrum $\mathbb{S}$, and A the Eilenberg-Mac Lane spectrum $H\mathbb{F}_P$. Then T-complete is the same as p-complete, and the homotopy groups of $W = H\mathbb{F}_P \wedge H\mathbb{F}_p$ form the dual of the Steenrod algebra. Thus the descent spectral sequence reproduces the Adams spectral sequence.

7.3. **Goodwillie's Calculus of Functors.** Let $\mathscr{C}$, $\mathscr{D}$ be two $\mathcal{T}op$-categories tensored and cotensored over $\mathcal{T}op$ and $F : \mathscr{C} \to \mathscr{D}$ be a continuous functor. The idea of Goodwillie's calculus is analogous to the usual calculus, we are going to indicate how we can "approximate" the functor F by "polynomial functors" $P_n F : \mathscr{C} \to \mathscr{D}$.

So, first we need to define what are polynomial functors. We will not define them precisely, but give and idea of its definition. The idea is that a functor P is a *polynomial functor* of degree $\leq n$ if it satisfies some conditions on its values on $(n+1)$-fold coproducts. Note that a usual function f is said to be polynomial of degree $\leq n$ if $f(x_0 + \cdots + x_n)$ can be computed by its values on the subsets of $\{x_0, \ldots, x_n\}$.

Example 7.4. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be of degree 1 if it takes homotopy push out squares to homotopy pull back squares. In particular,

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y \arrow[d] & & F(*) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ X \arrow[r] & X \sqcup Y \arrow[r, phantom, "\overset{F}{\longmapsto}"] & & F(X) \arrow[r] & F(X \sqcup Y) \end{tikzcd}$$

where the square on the right is a homotopy pull back. Note that this condition is analogous to the Mayer-Vietoris property in the sense that, if we have a pushout

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ U \arrow[r, hook] & X, \end{tikzcd}$$

then we have a long exact sequence

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to \cdots$$

In this case, the functor $H_\bullet$ is given by the functor $(\pi_\bullet \circ F)$, where $F = \Omega^\infty (HE \wedge X)$, where Ω^∞ denotes the infinite loop space, HE denotes the Eilenberg-MacLane space and $\wedge$ denotes the smash product. Hence degree 1 polynomial functors generalize homology theories.

Example 7.5. An example of degree 2 polynomial functor is the following. Take a space X, consider its smash product $X \wedge X$ and then the infinite loop space of the infinite spectra $\Sigma^\infty\Omega^\infty (X \wedge X)$. Then we have a functor

$$\Sigma^\infty\Omega^\infty(- \wedge -) : \mathcal{T}op_* \to \mathcal{T}op_*$$

From now on, lets fix the functor $F : \mathcal{T}op_* \to \mathcal{T}op_*$, which satisfies $F(*) = *$. In this case, F has a Taylor tower. A *Taylor tower* for F is an analogue of the Taylor series of a usual function. It consists of a tower

$$F \to \cdots \to P_nF \to P_{n-1}F \to \cdots \to P_1F \to P_0F = *,$$

where $P_nF : \mathcal{T}op_* \to \mathcal{T}op_*$ is a polynomial functor of degree n, which can be thought of as a degree n polynomial approximation of the functor F at the point $*$. Moreover, P_nF is universal in the sense that: there exists a natural transformation $\iota_n : F \to P_nF$, such that, for any functor G and any natural transfornation $\phi : F \to G$, there exists a unique natural transformation $\varphi : P_nF \to G$, such that the following diagram commutes.

$$\begin{tikzcd} F \arrow[rr, "\iota_n"] \arrow[dr, "\phi"'] & & P_nF \arrow[dl, "\varphi"] \\ & G & \end{tikzcd}$$

Theorem 7.6 (Goodwillie)**.** *Given F, there exist conditions on X, such that*

$$F(X) \simeq \text{holim } P_nF(X).$$

This theorem is usually true for sufficiently highly connected spaces X.

Example 7.7. Let $F = id : \mathcal{T}op_* \to \mathcal{T}op_*$. The identity functor is not polynomial of degree n for any n, since it does not take push outs to pull backs. So what are P_nid? These functors are actually hard to write down explicitly. What we are going to do is to describe the fibers of the maps on the tower and the first page of the spectral sequence that arises. Note however that, if X is simply connected, then $X \simeq \text{holim } P_nid(X)$, for any n.

Now, lets define the analogue to the terms on the Taylor series expansion of a usual function. For each n, define the n-th *Taylor coefficient*

$$D_nF = \text{hofib}(P_nF \to P_{n-1}F).$$

This is a *homogeneous* degree n functor, i.e. $P_{n-1}(D_nF)$ is trivial.

Theorem 7.8 (Goodwillie). *For any functor F, there exist $\partial_n F$, such that*

$$D_n F(X) = \Omega^\infty \left(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n}\right)_{h\Sigma_n},$$

where $\wedge n$ denotes the n-th smash power, $\Sigma^\infty X$ denotes the spectrum associated to X, Ω^∞ denotes the infinite loop space associated to the spectrum $\left(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n}\right)$, which admits a Σ_n-action and $h\Sigma_n$ denotes its homotopy coinvariants.

The theorem above describes the analogue of the n-th term of the Taylor series of F. But how do we combine these terms up in order to recover the Taylor tower of F from the Taylor coefficients $\partial_n F$ (and some additional information)?

Example 7.9.

$$\partial_n id = \bigvee_{(n-1)!} S^{1-n}.$$

Note that, in the above theorem, Σ_n acts on $\partial_n F$. To explain this action, we will compute its homology and explain how Σ_n acts on it. First take the free Lie algebra in n generators and denote it by $\mathcal{L}$. Now, consider the subspace $\mathcal{Lie}(n) \subseteq \mathcal{L}$ spanned by the elements of $\mathcal{L}$, where each generator of the free Lie algebra appears only once.

Example 7.10. $\mathcal{Lie}(3)$ is a 2-dimensional subspace of $\mathcal{L}$ spanned by

$$\{[\mathsf{x}, [\mathsf{y}, \mathsf{z}]], [\mathsf{y}, [\mathsf{x}, \mathsf{z}]]\}.$$

In general, $\mathcal{Lie}(n)$ has dimension $(n-1)!$ and its generators are obtained fixing the last generator and permuting the others.

Now, what does it have to do with descent theory? To explain the relation lets change the focus from $F : \mathcal{Top}_* \to \mathcal{Top}_*$ to functors $F : \mathcal{Top}_*^{fin} \to \mathit{Spectra}$ satisfying $F(*) = *$. These functors still have Taylor towers and Taylor coefficients.

Definition 7.11. A *symmetric sequence* of spectra is a sequence

$$A_1, A_2, A_3 \ldots,$$

where A_n is a spectrum with a Σ_n-action.[2]

Example 7.12. The sequence $\{\partial_n F\}_n$ is a symmetric sequence.

The problem of recovering the Taylor tower from the Taylor coefficients is a descent problem. Lets state the complete picture.

Let $\mathscr{C}$ be the category $\left[\mathcal{Top}_*^{fin}, \mathit{Spectra}\right]$ of functors from finite cell complexes to spectra and $\mathscr{D}$ be the category of symmetric sequences.

Theorem 7.13 (Arone-Ching). *The functor $\partial_\bullet : \mathscr{C} \to \mathscr{D}$ has a right adjoint*

$$\Phi A : X \longmapsto \hom_{\mathscr{D}}(\partial_\bullet R_X, A),$$

where $X \in \mathcal{Top}_*^{fin}$, R_X is a representable functor

$$\begin{aligned} R_X :\ \mathcal{Top}_* &\longrightarrow \text{Spectra} \\ Y &\longmapsto \Sigma^\infty \hom(X, Y). \end{aligned}$$

[2]But with no relation between them.

Now we can apply the descent picture

$$\begin{tikzcd} & \mathscr{D}_{\mathbb{K}} \arrow[d, "forget"] \\ \mathscr{C} \arrow[ur, dotted] \arrow[r, shift left, "\partial_\bullet"] & \mathscr{D}, \arrow[l, shift left, "\Phi"] \end{tikzcd}$$

where $\mathbb{K} = \partial_\bullet \Phi$. hence, for any functor $F : \mathcal{T}op_*^{fin} \to Spectra$, we have a $\mathbb{K}$-coalgebra structure on the Taylor coefficients $\partial_\bullet F$.

References

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