Constructing Spectral Sequences

Niles Johnson
http://www.nilesjohnson.net/ss-construction.html

April, 2012

We give a schematic diagram for the construction of a spectral sequence for the homology of a filtered complex. These slides assume that the reader is familiar with the basic structure of spectral sequences. Our goal is to give a visual aid to complement the standard construction. Our specific notation follows §5.4 of

Weibel, *An introduction to homological algebra*. Cambridge studies in advanced mathematics, **38**. 1994.

As Weibel does, we drop the complementary degree q for readability.

The notes here are a companion to the slide presentation available at the URL above. Those slides present the diagram in three layers, with more information appearing on each new layer. Here, we use italic text to add explanatory remarks which do not appear on the slides.

Let C be a filtered chain complex. For each n:

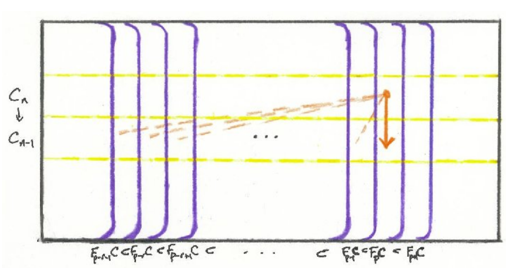
$$\cdots \subset F_{p-1}C_n \subset F_pC_n \subset F_{p+1}C_n \cdots$$

$$\eta_p : F_pC \to F_pC/F_{p-1}C$$

We say that $c \in F_pC_n$ "falls r steps in the filtration" if $d(c) \in F_{p-r}C_{n-1}$. We define A_p^r to be the submodule in F_p of elements which fall r steps in the filtration:

$$A_p^r := \left\{ c \in F_pC \,\middle|\, d(c) \in F_{p-r}C \right\}$$

Here we have a picture of a filtered complex. The differential takes C_n to C_{n-1}, and does not raise filtration degree. It may, however, lower *filtration degree.*



The approximate cycles Z_p^r *are defined as the image of* A_p^r *modulo filtration* $p-1$. *The approximate boundaries* B_p^r *are those elements which have fallen* $r-1$ *steps to land in filtration* p, *and we take these also modulo filtration* $p-1$.

$Z_p^r := \eta_p A_p^r$

$B_p^r := \eta_p dA_{p+r-1}^{r-1}$, $\qquad B_{p-r}^{r+1} := \eta_{p-r} dA_p^r$, $\qquad B_{p-r-1}^{r+2} := \eta_{p-r-1} dA_p^{r-1}$

Z_p^r and B_p^r induce filtrations on $E_p^0 := F_p/F_{p-1}$ Rotate the picture to view this filtration ...

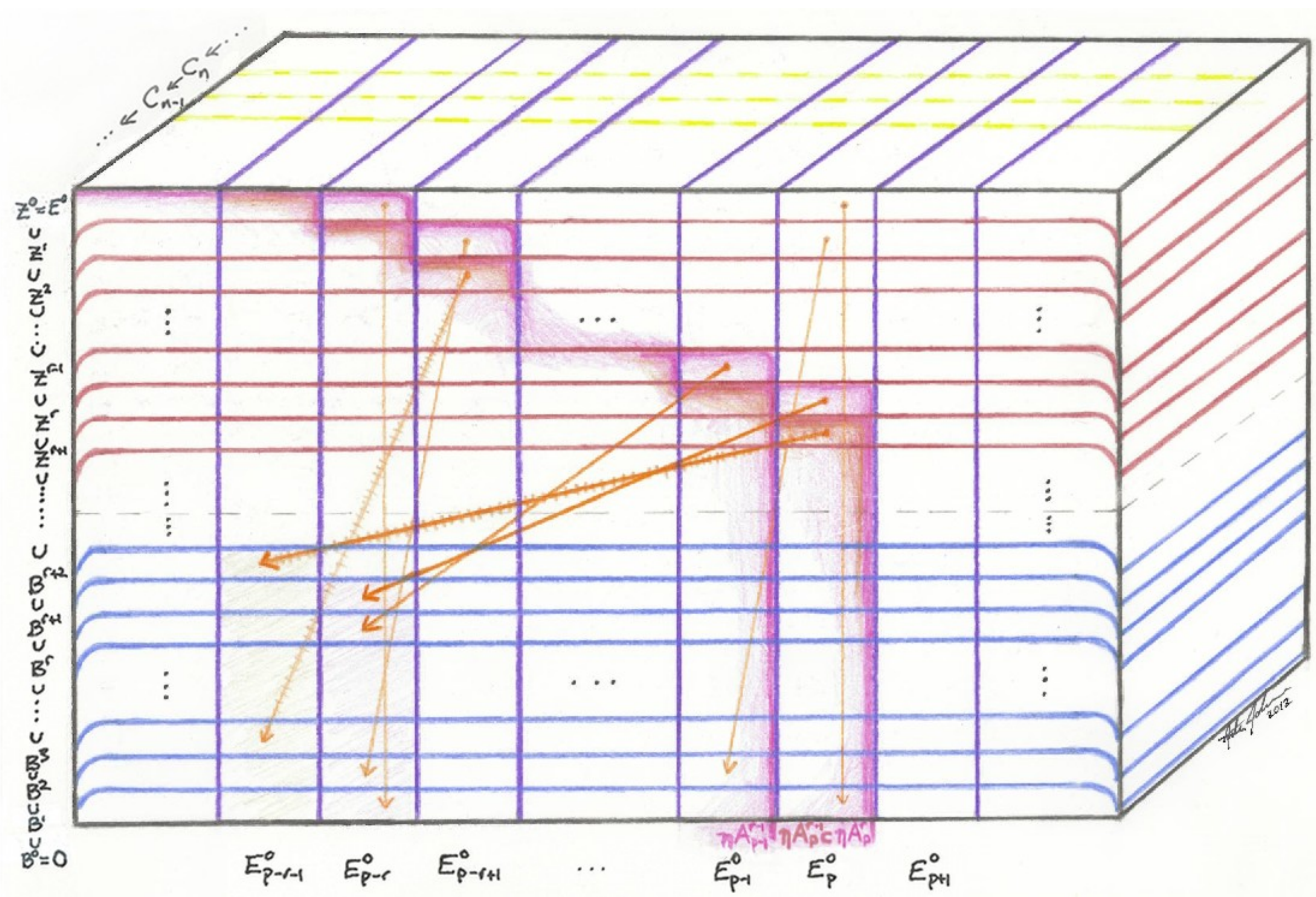


Figure 1: *Diagram to illustrate the construction of the spectral sequence. The filtration induced by the* Z_p^r *and* B_p^r *is shown horizontally, and the arrows show the maps*

$$Z_p^r/Z_p^{r+1} \to B_{p-r}^{r+1}/B_{p-r}^r$$

induced by the differential of C. *The shaded regions mark the submodules* A_p^r *in the various filtration quotients.*

1 Outline of the construction

Now we outline the construction, referring to the diagram shown above.

- Define

$$E_p^r := Z_p^r / B_p^r$$

This is a block in column p which gets smaller as r increases.

- Note

$$\begin{array}{ccc} A_p^r & \xrightarrow{\eta_{p-r}\, d} & B_{p-r}^{r+1} \\ \cup\uparrow & \overset{0}{\nearrow} & \\ A_p^{r+1} & \xrightarrow{\eta_{p-r-1}\, d} & B_{p-r-1}^{r+2} \end{array}$$

Working modulo filtration $p-r-1$, the image of A_p^{r+1} is zero because these elements actually fall $r+1$ steps in the filtration and land in $F_{p-r-1} \subset F_{p-r}$. However, this image is potentially non-zero when working modulo filtration $p-r-2$, and surjects onto B_{p-r-1}^{r+2}. The hatched differentials in the diagram are taken modulo F_{p-r-2} instead of F_{p-r-1}.

- Lemmas (below):

$$d : Z_p^r / Z_p^{r+1} \xrightarrow{\cong} B_{p-r}^{r+1} / B_{p-r}^r$$

This is the technical key to the construction. We explain how to finish the construction and then return to this point.

- Define

$$d_p^r : Z_p^r / B_p^r \to Z_p^r / Z_p^{r+1} \cong B_{p-r}^{r+1} / B_{p-r}^r \to Z_{p-r}^r / B_{p-r}^r$$

This is the definition of the rth differential. It takes the rth block in column p, kills everything except the topmost box, takes that to its isomorphic image under the differential, and then includes as the bottommost box in the rth block in column $p-r$.

With this description it is easy to identify the kernel and image of d^r, which we now do:

- Observe:

$$ker(d_p^r) = Z_p^{r+1} / B_p^r$$
$$im(d_{p+r}^r) = B_p^{r+1} / B_p^r$$

- Conclude

$$E_p^{r+1} \cong ker(d_p^r) / \, im(d_{p+r}^r)$$

This finishes the construction: We have described the pages of the spectral sequence, defined the differential, and shown that the homology at position p on page r is the term at position p on page $r+1$.

2 Lemmas

Now we describe the two lemmas which identify various quotients in E^0 with quotients in C. This is done by careful analysis of the terms

$$\begin{array}{ccc} A_{p-1}^{r-1} & \subset & A_p^r \\ \cup & & \cup \\ A_{p-1}^r & \subset & A_p^{r+1} \end{array}$$

The shaded regions on the diagram show the images of the A_p^r in various filtration quotients. To verify each isomorphism, we recommend first observing how the two sides of each equation describe the same region on the diagram. We repeatedly apply the following two general results from group theory:

$$\frac{G/K}{H/K} \cong G/H \qquad (*)$$

$$\frac{I+J}{J} \cong \frac{I}{I \cap J} \qquad (**)$$

Lemma 1

$$A_p^r \cap F_{p-1} \cong A_{p-1}^{r-1} \tag{1}$$

$$Z_p^r \cong A_p^r / A_{p-1}^{r-1} \tag{2}$$

$$Z_p^r / B_p^r \cong \frac{A_p^r + F_{p-1}}{dA_{p+r-1}^{r-1} + F_{p-1}} \cong \frac{A_p^r}{dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}} \tag{3}$$

For Equation (3), the first isomorphism follows from $(*)$ by taking $K = F_{p-1}$ and the second follows from $(**)$ by taking $J = dA_{p+r-1}^{r-1} + F_{p-1}$ and using (1).

Lemma 2

$$Z_p^r / Z_p^{r+1} \cong \frac{A_p^r}{A_{p-1}^{r-1}} \Big/ \frac{A_p^{r+1}}{A_{p-1}^r} \cong \frac{A_p^r}{A_p^{r+1} + A_{p-1}^{r-1}} \tag{4}$$

$$B_{p-r}^{r+1} / B_{p-r}^r \cong \frac{dA_p^r}{dA_p^{r+1} + dA_{p-1}^{r-1}} \tag{5}$$

Equation (4) follows by first using $(**)$ (in "reverse") on Z_p^{r+1} with $I = A_p^{r+1}$ and $J = A_{p-1}^{r-1}$. Second, use $(*)$ with $K = A_{p-1}^{r-1}$. Equation (5) is identical, but using the images of these groups under the differential.

The isomorphism

$$Z_p^r / Z_p^{r+1} \cong B_{p-r}^{r+1} / B_{p-r}^r$$

follows by using the definition of the terms on the right hand side of Lemma 2 to verify directly that the kernel and image of the map induced by the differential are zero.