Euler Characteristic

An introduction to algebraic topology

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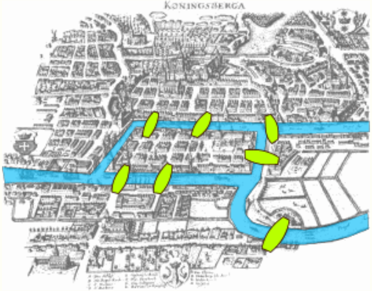
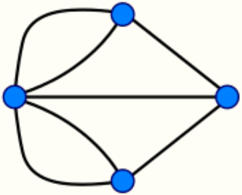
Abstract

This talk introduces Euler characteristic for plane graphs and convex polyhedra. We describe several puzzles that can be solved by using Euler characteristic. At the end we point out a few connections to other topics in algebraic topology.

Königsburg Bridges

Puzzle: Is there a path through the city that crosses each bridge exactly once?



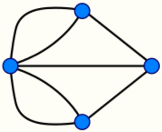


Abstraction: For this puzzle, distances in the city are not relevant. Space between bridges is also irrelevant for this. Consider an abstract *planar graph*.

https://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg

Königsberg Bridges

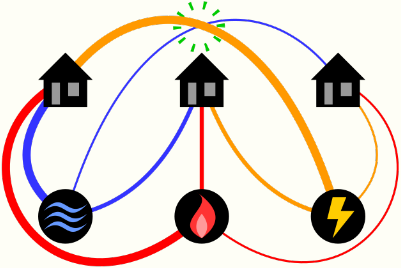
Observation: If a graph can be traversed with a path that only crosses each edge once, then every vertex except the start/ending will have to touch an even number of edges (“in” and “out” edges).



Solution: There is *no path* that crosses each of the seven bridges exactly once! Each of the four land masses has an odd number of bridges, so *each land* would have to be either the beginning or ending of the path.

Three Utilities

Puzzle: Is there a way to connect each house with each of the three utilities, without crossing lines?

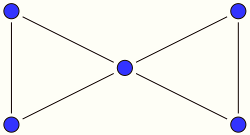


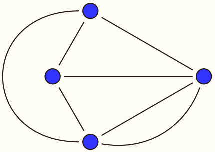
Abstraction: Drawing a graph in the plane consists of vertices, connected by edges, and separates the plane into different regions called *faces*.

https://en.wikipedia.org/wiki/Three_utilities_problem

Planar Graphs

Definition: A graph is called *planar* if it can be drawn in the plane (no edges crossing). Note: the same graph might be drawn different ways in the plane; that's ok.





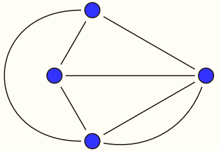
Observations:

- Adding an edge to a plane graph always increases the number of faces by *exactly one*
- The minimum number of edges for a *connected* plane graph is one fewer than the number of vertices.

Planar Graphs

Euler Characteristic of Plane Graph: Consider a connected planar graph, drawn in the plane. Let V be its number of vertices, E its number of edges, and F the number of faces (regions bounded by the edges). Then

$$V - E + F = 1.$$

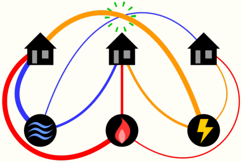


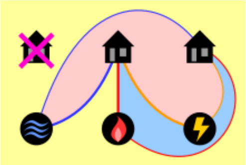
Example: $V = 4$, $E = 7$, $F = 4$.

Why does this happen??!! (Proof): For the straight line graph (minimal number of edges), we have $V - E = 1$ and $F = 0$. Then every new edge we add also adds one face, so they cancel out.

Revisiting the Three Utilities

Puzzle: Is there a way to connect each house with each of the three utilities, without crossing lines?





Solution: If there were a way to draw the three utilities graph in the plane, then it would have to have $V = 6$, $E = 9$, and $F = 4$.

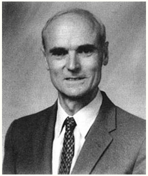
This leads to a contradiction because in this graph every face has at least *four* edges, so (after some more thinking) any planar drawing of this graph should have $F + 1 \leq \frac{E}{2}$. But 5 is *greater* than $\frac{9}{2}$.

Note: The explanation we skipped works by also counting the outer, unbounded face. We will talk more about that soon.

Some of the people involved







Francesco Maurolico (early 1500s) (~Columbus)
https://en.wikipedia.org/wiki/Francesco_Maurolico

Leonhard Euler (mid 1700s) (~U.S. Independence)
https://en.wikipedia.org/wiki/Leonhard_Euler

Emmy Noether (mid 1800s) (~Jefferson, Lewis & Clark)
https://en.wikipedia.org/wiki/Emmy_Noether

Alicia Boole Stott (late 1800s, early 1900s) (~U.S. Civil War)
https://en.wikipedia.org/wiki/Alicia_Boole_Stott

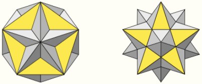
Harold Coxeter (early 1900s) (~Einstein)
https://en.wikipedia.org/wiki/Harold_Scott_MacDonald_Coxeter

Euler Characteristic of Polyhedra

Convex Polyhedra: 3D solids formed by vertices, straight edges, flat faces, that are *not* self-intersecting and whose interiors are *convex*.

(General definition of polyhedron much more complicated.)

Convex: 

Non-Convex:  *(self-intersecting face)*

Euler Characteristic for Convex Polyhedra:

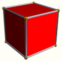
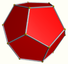
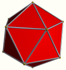
$$\chi = V - E + F = 2$$

(One more than Euler characteristic for plane graphs.)

https://en.wikipedia.org/wiki/Polyhedron

https://en.wikipedia.org/wiki/Euler_characteristic

Euler Characteristic of Platonic Solids

Name	Image	Vertices	Edges	Faces	V – E + F
Tetrahedron		4	6	4	2
Cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2
Any Others?	???	V	E	F	2

Euler Characteristic of Platonic Solids

Puzzle: Are there any other Platonic solids?

Definition: A Platonic solid has regular n-gon faces, for some n, and has k faces meeting at each vertex.

Observation: In a Platonic solid, $E = \frac{nF}{2}$, so $F = \frac{2E}{n}$. Also, $V = \frac{nF}{k} = \frac{2E}{k}$. So

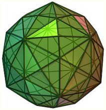
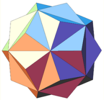
$$2 = \frac{2E}{k} - E + \frac{2E}{n} = \left(\frac{1}{k} - \frac{1}{2} + \frac{1}{n}\right) \cdot (2E).$$

Check: The only possible solutions are

$$(n, k) = (3,3);\ (4,3);\ (3,4);\ (5,3);\ (3,5).$$

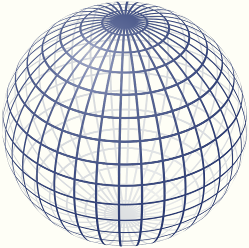
There are no more Platonic solids!

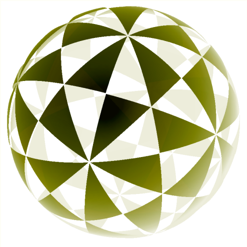
Euler Characteristic of Other Polyhedra

Name	Image	Vertices	Edges	Faces	V – E + F
d120 (Disdyakis Triacontahedron)		62	$\frac{3\cdot120}{2} = 180$	120	2
Small Triambic Icosahedron (not convex)		32	90	60	2

Euler Characteristic of the Sphere

Really, Euler characteristic $\chi = 2$ is a property of the sphere (in 3-dimensions). A convex polyhedron basically determines a graph on the surface of the sphere.



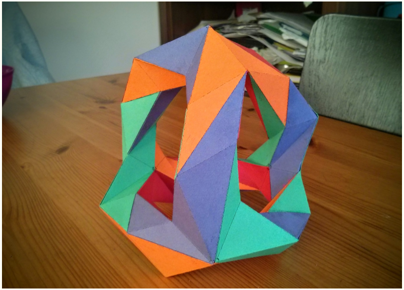


Euler Characteristic of *Other Things*

Rectangular Frame

$V = 16, E = 32, F = 16$

$\chi = V - E + F = 0$



Klein Quartic

$V = \frac{24 \cdot 7}{3} = 56,$

$E = \frac{120 \cdot 3}{2} = 180,$

$F = 24 \cdot 5 = 120$

$\chi = V - E + F = -4$

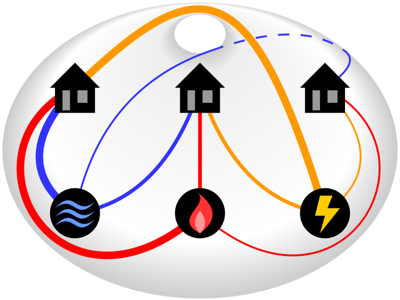
https://github.com/timhutton/klein-quartic

Euler Characteristic of *Other Things*

Rectangular Frame

$V = 16$, $E = 32$, $F = 16$

$\chi = V - E + F = 0$



Three Utilities on a Torus

$V = 6$, $E = 9$, $F = 3$

$\chi = V - E + F = 0$

Euler Characteristic of *Other Things*

Euler characteristic is a *topological invariant*. It measures something about the basic shape, not depending on lengths, areas, etc.
Different subdivisions into vertices, edges, and faces all result in the same Euler characteristic.

There is a generalization of Euler characteristic for higher-dimensional shapes too.

$$\chi = V - E + F_2 - F_3 + F_4 - F_5 + \cdots$$

where F_k is the number of regions of dimension k.
(So $V = F_0$ and $E = F_1$.)

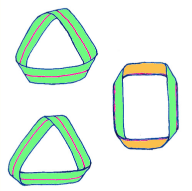
Next Steps in Algebraic Topology

Genus of surfaces

χ = 2 – 2(genus)

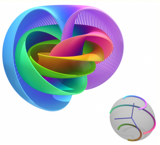
https://en.wikipedia.org/wiki/Genus_(mathematics)

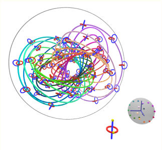
Knots and links



https://en.wikipedia.org/wiki/Linking_number

Higher-dimensional properties





https://en.wikipedia.org/wiki/Hopf_fibration

The End

Thank You!