The Adams-Novikov E_2-term for $Q(2)$ at the prime 3

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Motivation and construction of $Q(2)$

Computing the ANSS E_2-term for $Q(2)$

Some applications and directions

The spectrum $Q(2)$

Theorem (Behrens 2006)

(a) *At the prime 3, there exists an E_∞-ring spectrum $Q(2)$, built using degree 2 isogenies of elliptic curves, with the property that*

$$DQ(2) \xrightarrow{D\eta} S_{K(2)} \xrightarrow{\eta} Q(2)$$

is a cofiber sequence.

(b) *The Adams-Novikov E_2-term for $Q(2)$ is the target of a double cochain complex spectral sequence.*

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Let $p \geqslant 5$*.*

(a) $Q(N)$ *detects the homotopy elements* $\alpha_{i,j}$ *and* $\beta_{i/j,k}$*.*

(b) *There is a 1-1 correspondence between additive generators* $\beta_{i/j,k} \in \mathrm{Ext}^2(BP_*)$ *and modular forms* $f_{i/j,k}$ *of weight* $2i(p^2-1)$ *satisfying certain congruence conditions.*

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Conjecture

This holds at $p = 3$. [Different techniques are required.]

$Q(2)$ as a modular interpretation

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At $p = 2$, $S_{K(1)} \simeq J \to KO_2^\wedge \xrightarrow{\psi^3 - 1} KO_2^\wedge$.

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$$S_{K(2)} \to X_1 \to X_2 \to X_3 \to X_4 \to X_5$$

which refines to a 4-stage tower of fibrations with $S_{K(2)}$ at the top.

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Remark

The X_i above are wedges of suspensions of $E_2^{hG_{24}} \simeq tmf$ and $E_2^{hD_8} \simeq tmf_0(2)$.

The definition of $Q(2)$

Motivated by the GHMR resolution, and by work of Mahowald-Rezk on a map

$$tmf \xrightarrow{\text{“}\psi^3-1\text{”}} tmf_0(3)$$

at the prime 2, Behrens constructed $Q(2)$ as a semi-cosimplicial spectrum, as follows.

Definition

$$Q(2) = \mathrm{Tot}\left[tmf \Rightarrow tmf \vee tmf_0(2) \Rrightarrow tmf_0(2) \right]$$

Algebraic underpinnings of $Q(2)$

The key algebraic object is the Hopf algebroid (B, Γ), where

$$B = \mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}]/(\Delta = q_4^2(16q_2^2 - 64q_4)),$$
$$\Gamma = B[r]/(r^3 + q_2 r^2 + q_4 r).$$

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Proposition

The ANSS for tmf takes the form

$$\mathrm{Ext}^* := \mathrm{Ext}^*_\Gamma(B, B) \Rightarrow \pi_* tmf$$

while the ANSS for $tmf_0(2)$ collapses at E_2 to yield

$$\pi_{2k} tmf_0(2) = B_k.$$

Setting up the ANSS

The semi-cosimplicial diagram above topologically realizes

$$\mathcal{M} \Leftarrow \mathcal{M} \coprod \mathcal{M}_0(2) \Lleftarrow \mathcal{M}_0(2).$$

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Proposition (Behrens)

(a) *The ANSS for $Q(2)$ is*

$$E_2^{s,t} = \mathbb{H}^{s,t}(\mathcal{M}_\bullet) \Rightarrow \pi_{2t-s} Q(2).$$

(b) *The hypercohomology SS converging to this E_2-term is the double complex SS for $C^{*,*}$, given by*

$$C^*(\Gamma) \to \overline{C}^*(\Gamma) \oplus B \to B \to 0.$$

The main theorem

Theorem (L.)

$H^k(\mathrm{Tot}\, C^{*,*}) = M_k \oplus N_k$, where

$$M_k = \begin{cases} \mathbb{Z}_{(3)}\{1_{MF}\}, & k = 0, 1 \\ \mathrm{Ext}^k \oplus \mathrm{Ext}^{k-1}, & k \geqslant 1 \end{cases}$$

and

$$N_k = \begin{cases} \widetilde{N} \oplus \mathbb{Z}_{(3)}\{\alpha\} & k = 1 \\ \mathrm{coker}\, d_2 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

Here, d_2 is the only nontrivial differential on the E_2-page of the double complex SS, $\widetilde{N}$ is a countable direct sum of cyclic $\mathbb{Z}_{(3)}$-modules, and Ext^* is torsion for $* \geqslant 1$ (T. Bauer).

The rational homotopy of $Q(2)$

Theorem (Behrens)

The rational homotopy of $Q(2)$ is

$$\pi_k Q(2) \otimes \mathbb{Q} = \begin{cases} \bigoplus_n \mathbb{Q}, & k = -2 \\ \mathbb{Q}\{1_{MF}\} \oplus \bigoplus_n \mathbb{Q}, & k = -1 \\ \mathbb{Q}\{1_{MF}\}, & k = 0 \\ 0, & \textit{otherwise.} \end{cases}$$

The double cochain complex

Expanding $C^{*,*}$ yields

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\Gamma \otimes \Gamma & \xrightarrow{\Phi} & \Gamma \otimes \Gamma & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
d \uparrow & & -d \uparrow & & \uparrow & & \\
\Gamma & \xrightarrow{\Phi} & \Gamma & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
d \uparrow & & -d \oplus 0 \uparrow & & \uparrow & & \\
B & \xrightarrow{\Phi} & B \oplus B & \xrightarrow{\Psi} & B & \longrightarrow & 0 \cdots
\end{array}$$

The ring of invariants of (B, Γ)

Lemma

$$\mathrm{Ext}^0 = \mathbb{Z}_{(3)}[c_4, c_6, \Delta, \Delta^{-1}]/(1728\Delta = c_4^3 - c_6^2) =: MF$$

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Remark

The E_1-page of the double complex SS becomes

$$\begin{array}{ccccccc} \vdots & & \vdots & & & & \\ \mathrm{Ext}^1 & \xrightarrow{0} & \mathrm{Ext}^1 & \xrightarrow{0} & 0 & \longrightarrow & 0\cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ MF & \xrightarrow{\Phi} & B \oplus MF & \xrightarrow{\Psi} & B & \longrightarrow & 0\cdots \end{array}$$

The maps Φ and Ψ

Let C denote the complex $MF \xrightarrow{\Phi} B \oplus MF \xrightarrow{\Psi} B$. The maps Φ and Ψ are explicitly known.

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Example

The map $\psi_d : tmf_0(2) \to tmf_0(2)$ realizes $\psi_d : \mathcal{M}_0(2) \to \mathcal{M}_0(2)$ which, given a $\mathbb{Z}_{(3)}$-algebra T, sends an elliptic curve E over T to E/H; the corresponding effect on Weierstrass equations determines that $\psi_d : B \to B$ is defined by

$$q_2 \mapsto -2q_2, \quad q_4 \mapsto -4q_4 + q_2^2$$

and $\Psi = (\psi_d^* + 1) \oplus \gamma$ for $\gamma : MF \to B$.

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Notation

Let $\Phi(x) = (f(x), g(x))$ and $\Psi(y, z) = h(y) + k(z)$.

A two-stage filtration of C

We filter C as follows:

$$F^0 = C,$$
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F^2 &= 0.
\end{aligned}
$$

This yields a SES of complexes

$$0 \to C' \to C \to C'' \to 0$$

where $C' = (0 \to B \xrightarrow{h} B)$ and $C'' = F^1$. We obtain a LES in cohomology

$$H^0 C \hookrightarrow \ker g \xrightarrow{\delta^0} \ker h \to H^1 C \to \operatorname{coker} g \xrightarrow{\delta^1} \operatorname{coker} h \twoheadrightarrow H^2 C$$

Example

If $x \in MF$ is a modular form of weight k, then

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Proposition (L.)

coker g *has as a direct summand*

$$\bigoplus_x \mathbb{Z}/3^{\mu_3(k)+1}\mathbb{Z}$$

where x runs through elements of nonzero weight in an additive basis for MF.

A conjecture

Conjecture

Let F be the map between the Adams-Novikov E_2-terms for the ($K(2)$-local) sphere and $Q(2)$ induced by the unit map of $Q(2)$. Then F detects the algebraic alpha and beta families.

The End

- Preprint in progress.
- See also *On the homotopy of* $Q(3)$ *and* $Q(5)$ *at the prime 2* [Behrens-Ormsby]

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Thank you!