On certain tertiary homotopy operations

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Outline

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- Toda brackets and other secondary topological operations

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- secondary operations in a 2-category with zeros

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- tertiary topological operations

Homotopy groups of spheres

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  "sphere of origin $>$ stem $+1$"
- in which case by a Theorem of Freudenthal
  $\pi_{n+k} S^n \cong \pi_{n+k+1} S^{n+1} \cong \pi_{n+k+2} S^{n+2} \cong \cdots$
  defining "the stable k-stem" π_k^S

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- occasionally stable stems are trivial:

$$\pi_4^S = 0, \ \pi_5^S = 0, \ \pi_{12}^S = 0, \ \pi_{29}^S = 0$$

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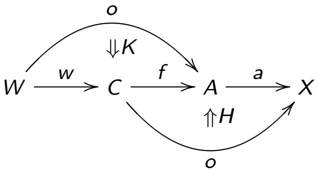
- Toda brackets (secondary compositions) $\{\alpha, \beta, \gamma\}$

  $S^p \xrightarrow{\gamma} S^m \xrightarrow{\beta} S^k \xrightarrow{\alpha} S^n$

  $\{\alpha, \beta, \gamma\} \subset \pi_{p+1} S^n$ is defined if $\alpha \circ \beta \simeq 0$ and $\beta \circ \gamma \simeq o$

Toda bracket operation



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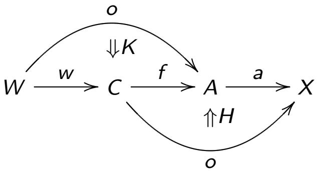

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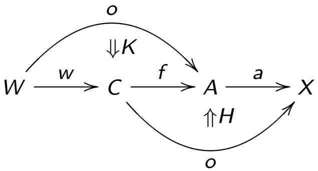

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- indeterminacy of $\{a, f, w\}$ is $a \circ \pi(\Sigma W, A) + \pi(\Sigma C, X) \circ \Sigma w$
- explicit definition:

  for $z \in W$, $[z, t] \mapsto \begin{cases} H(w(z), 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ aK(z, 2-2t), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$

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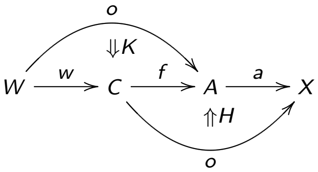

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- Toda (1952) showed that $\nu' \in \{\eta_3, 2\iota_4, \eta_4\} \subset \pi_6 S^3 = \mathbb{Z}/12$ has order 4, with indeterminacy $\mathbb{Z}/2$. This was the first nontrivial Toda bracket computed.

- if $a \circ b \simeq o$ and $b \circ c \simeq o$ then generally

$$f \circ \{a, b, c\} \subset \{f \circ a, b, c\}$$
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  Toda's Lemma: if $a_0 \circ a_1 \simeq o$, $a_1 \circ a_2 \simeq o$, $a_2 \circ a_3 \simeq o$ then

$$a_0 \circ \{a_1, a_2, a_3\} = -\{a_0, a_1, a_2\} \circ Ea_3$$

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- the equality in Toda's Lemma gives rise to a *quaternary Toda bracket* $\{a_0, a_1, a_2, a_3\}$. This is a tertiary operation (raising stem dimension by 2). Toda made mention of the operation in an announcement (C.R. Paris 1955); it was developed by Ôguchi (1963). The definition requires a coherence condition and is obscure and technically difficult.

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- the quaternary Toda bracket is needed in the 9-stem to describe one of the generators in

$$\pi_{12} S^3 = (\mathbb{Z}/2)^2 = \{\mu_3\} \oplus \{\eta_3 \circ \varepsilon_4\}.$$

Namely, $\mu_3 \in \{\eta_3, E\nu', 8\iota_7, \nu_7\} = \mu_3 + \{\eta_3 \circ \varepsilon_4\}$.

EHP sequence

The following sequence is called the *EHP* sequence

$$\cdots \longrightarrow \pi_k S^n \xrightarrow{E} \pi_{k+1} S^{n+1} \xrightarrow{H} \pi_{k+1} S^{2n+1} \xrightarrow{P} \pi_{k-1} S^n \xrightarrow{E} \cdots$$

where

E = suspension homomorphism

H = Hopf invariant

P is related to the Whitehead product

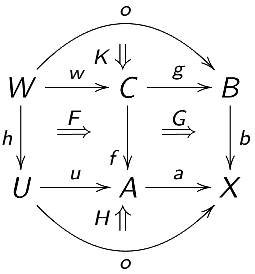
This sequence is

- exact if n is odd and $k < 3n - 3$

and

- exact on 2-primary components if $k > 2n$.

The box bracket operation (Hardie-Marcum-Oda, 2001)



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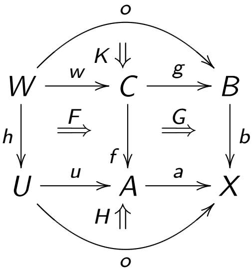

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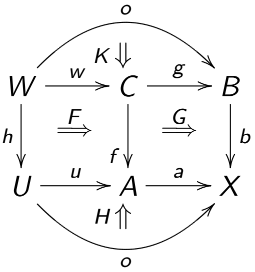

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- the Toda brackets $\{a, u, h\}$, $\{a, f, w\}$ and $\{b, g, w\}$ need not be defined!!

Some special box brackets

- $\mathcal{L}_2 \left(\begin{array}{ccccc} W & \xrightarrow{w} & C & \longrightarrow & * \\ \downarrow & & \downarrow f & & \downarrow \\ * & \longrightarrow & A & \xrightarrow[a]{} & X \end{array} \right) = -\{a, f, w\}$

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  Barratt (1963), Mimura (1964)

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  Mori(1971,stably), Hardie-Kamps-Marcum (1991)

Definition of a 2-category $\mathcal{C}$

- objects X

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  2-morphisms $X \overset{f}{\underset{g}{\Downarrow F}} Y$

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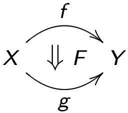
  2-morphisms 

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- **Example:** $\mathcal{T}op_*$

  objects: based topological spaces X

  1-morphisms: continuous based maps $X \xrightarrow{f} Y$

  2-morphisms: track classes of based topological homotopies

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- There is an associated homotopy category H$\mathcal{C}$
  $f, g\colon X \to y$ are *homotopic* if there exists an invertible 2-morphism $f \Rightarrow g$
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- The set of all invertible self 2-morphisms of a 1-morphism f will be denoted

$$\mathcal{A}_{\mathcal{C}}(f\colon X \to Y) := \left\{ X \underset{f}{\overset{f}{\Downarrow F}} Y \;\middle|\; F \text{ is invertible} \right\}$$

and is a group under vertical composition of 2-morphisms. The group identity is 1_f.

Suspension 2-functor

Let $\mathcal{C}$ be a 2-category with zeros and $\Sigma\colon \mathcal{C} \to \mathcal{C}$ a 2-functor preserving zeros. For each object X of $\mathcal{C}$ let there be assigned an invertible 2-morphism

$$X \underset{o}{\overset{o}{\Downarrow D_X}} \Sigma X$$

which is natural in X. Then for each pair of objects (X, Y) there is an induced function

$$\varphi = \varphi_{(X,Y)} \colon \mathbb{H}\mathcal{C}(\Sigma X, Y) \to \mathcal{A}_{\mathcal{C}}(o \colon X \to Y)$$

given by

$$\alpha \colon \Sigma X \to Y \quad \mapsto \quad X \underset{o}{\overset{o}{\Downarrow \alpha D_X}} Y$$

If φ is a bijection for all pairs (X, Y) then Σ is called a *suspension 2-functor*

The 2-sided matrix Toda bracket

Let $\mathcal{C}$ be a 2-category with zeros.

$$\dot{\square}\left(\begin{array}{ccccc} W \overset{w}{\to} & C & \overset{g}{\to} & B & \\ & f\downarrow & & \downarrow b & \\ & A & \underset{a}{\to} & X & \underset{s}{\to} Y \end{array}\right) \subset \mathcal{A}_{\mathcal{C}}(o : W \to Y)$$

It is defined if $a \circ f \simeq b \circ g$, $g \circ w \simeq o$ and $s \circ a \simeq o$. It consists of all 2-morphisms of the form

$$-(s \circ b)K + sFw + H(f \circ w) \colon o \Rightarrow o \colon W \to Y$$

for homotopies $H \colon o \Rightarrow s \circ a$, $F \colon a \circ f \Rightarrow b \circ g$ and $K \colon o \Rightarrow g \circ w$.

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for homotopies $H \colon o \Rightarrow s \circ a$, $F \colon a \circ f \Rightarrow b \circ g$ and $K \colon o \Rightarrow g \circ w$.

- If both $f \circ w \simeq o$ and $s \circ b \simeq o$ then

$$s \circ \left\{ \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix}, w \right\} = \dot{\underline{\square}} \left(\begin{array}{ccccc} W & \xrightarrow{w} & C & \xrightarrow{g} & B & & \\ & & f \downarrow & & \downarrow b & & \\ & & A & \xrightarrow[a]{} & X & \xrightarrow[s]{} & Y \end{array} \right) = \left\{ s, \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix} \right\} \circ w.$$

This is a generalization of Toda's Lemma.

Theorem

Let $\mathcal{C}$ be a 2-category with zeros. Let

$$\begin{array}{ccccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ {\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_3 & \xrightarrow[b_3]{} & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}$$

be a 3-box diagram of 1-morphisms in $\mathcal{C}$. Then $f_0 \circ \{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\} \circ f_3$ are each contained in the same double coset of the subgroups $b_1 \circ \mathcal{A}_{\mathcal{C}}(o : A_3 \to B_1)$ and $\mathcal{A}_{\mathcal{C}}(o : A_2 \to B_0) \circ a_3$ of the group $\mathcal{A}_{\mathcal{C}}(o : A_3 \to B_0)$. Consequently, letting this double coset be denoted by

$$\gamma \left(\begin{array}{ccccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ {\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_3 & \xrightarrow[b_3]{} & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array} \right) \subset \mathcal{A}_{\mathcal{C}}(o : A_3 \to B_0)$$

an operation associated to the 3-box diagram is obtained

Theorem

The inclusion

$$\square \left(\begin{array}{ccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & & \\ & & f_2 \big\downarrow & & \big\downarrow f_1 & & \\ & & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array} \right) \subset \gamma \left(\begin{array}{ccccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ f_3 \big\downarrow & & \big\downarrow f_2 & & \big\downarrow f_1 & & \big\downarrow f_0 \\ B_3 & \xrightarrow[b_3]{} & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array} \right)$$

holds in $\mathcal{A}_{\mathcal{C}}(o : A_3 \to B_0)$.

Theorem

The inclusion

$$\dot{\square}\left(\begin{array}{ccccc} A_3 & \overset{a_3}{\rightarrow} & A_2 & \overset{a_2}{\rightarrow} & A_1 & & \\ & & f_2\downarrow & & \downarrow f_1 & & \\ & & B_2 & \underset{b_2}{\rightarrow} & B_1 & \underset{b_1}{\rightarrow} & B_0 \end{array}\right) \subset \gamma \left(\begin{array}{ccccccc} A_3 & \overset{a_3}{\rightarrow} & A_2 & \overset{a_2}{\rightarrow} & A_1 & \overset{a_1}{\rightarrow} & A_0 \\ f_3\downarrow & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ B_3 & \underset{b_3}{\rightarrow} & B_2 & \underset{b_2}{\rightarrow} & B_1 & \underset{b_1}{\rightarrow} & B_0 \end{array}\right)$$

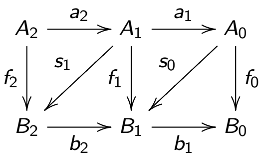
holds in $\mathcal{A}_{\mathcal{C}}(o : A_3 \rightarrow B_0)$.

- Inclusion can be proper; e.g., the 3-box diagram in $\mathrm{H}\mathcal{T}op_*$

$$\begin{array}{ccccccc} S^{21} & \xrightarrow{\nu_{15}^2} & S^{15} & \xrightarrow{\varepsilon_7} & S^7 & \xrightarrow{2\iota_7} & S^7 \\ \eta_{20}\downarrow & & \downarrow \nu_{12} & & \downarrow \eta_5^2 & & \downarrow E\nu' \\ S^{20} & \xrightarrow[\overline{\nu}_{12}]{} & S^{12} & \xrightarrow[\sigma''']{} & S^5 & \xrightarrow[\eta_4]{} & S^4 \end{array}$$

has $\dot{\square} = 0$ in $\pi_{22}S^4$ while
$\gamma = \{\eta_4 \circ \overline{\mu}_5, 2E\mu' \circ \sigma_{15}, E\nu' \circ \overline{\varepsilon}_7\} \cong (\mathbb{Z}/2)^3$.

Split 2-box diagrams



A homotopy commutative diagram of 1-morphisms with $a_1 \circ a_2 \simeq o$ and $b_2 \circ b_2 \simeq o$ in a 2-category $\mathcal{C}$ with zeros is said to be a *split* 2-*box diagram*.

Split 2-box diagrams

$$\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ {\scriptstyle f_2}\downarrow & \swarrow{\scriptstyle s_1} & {\scriptstyle f_1}\downarrow & \swarrow{\scriptstyle s_0} & \downarrow{\scriptstyle f_0} \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}$$

A homotopy commutative diagram of 1-morphisms with $a_1 \circ a_2 \simeq o$ and $b_2 \circ b_2 \simeq o$ in a 2-category $\mathcal{C}$ with zeros is said to be a *split 2-box diagram*.

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$$\square\left(\begin{array}{ccccc} A_2 & \overset{a_2}{\to} & A_1 & \overset{a_1}{\to} & A_0 \\ & & {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_0} \\ & & B_2 & \underset{b_2}{\to} & B_1 & \underset{b_1}{\to} & B_0 \end{array}\right) \subset \mathcal{L}_2\left(\begin{array}{ccccc} A_2 & \overset{a_2}{\to} & A_1 & \overset{a_1}{\to} & A_0 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_2 & \underset{b_2}{\to} & B_1 & \underset{b_1}{\to} & B_0 \end{array}\right)$$

is valid as subsets of $\mathcal{A}_{\mathcal{C}}(o\colon A_2 \to B_0)$

Theorem. Assume that $\mathcal{A}_{\mathcal{C}}(o\colon A_3 \to B_0)$ is abelian. Then the following hold.

- $$\gamma\left(\begin{array}{ccccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ {\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_3 & \xrightarrow[b_3]{} & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right) = \mathcal{L}_2\left(\begin{array}{ccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 \\ {\scriptstyle b_3\circ f_3}\downarrow & & \downarrow{\scriptstyle f_1\circ a_2} & & \downarrow{\scriptstyle f_0\circ a_1} \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right)$$

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- $$\gamma\left(\begin{array}{ccccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ f_3\big\downarrow & & \big\downarrow f_2 & & \big\downarrow f_1 & & \big\downarrow f_0 \\ B_3 & \xrightarrow[b_3]{} & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right) = \mathcal{L}_2\left(\begin{array}{ccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 \\ b_3\circ f_3\big\downarrow & & \big\downarrow f_1\circ a_2 & & \big\downarrow f_0\circ a_1 \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right)$$

- If

$$\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ f_2\big\downarrow & \overset{s_1}{\swarrow} & \big\downarrow f_1 & \overset{s_0}{\swarrow} & \big\downarrow f_0 \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}$$

is a split 2-box diagram then

$$\mathcal{L}_2\left(\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ f_2\big\downarrow & & \big\downarrow f_1 & & \big\downarrow f_0 \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right) = \gamma\left(\begin{array}{ccccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 & \xrightarrow{f_0} & B_0 \\ \mathrm{id}\big\downarrow & & \big\downarrow s_1 & & \big\downarrow s_0 & & \big\downarrow \mathrm{id} \\ A_2 & \xrightarrow[f_2]{} & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right).$$

Theorem

Consider a 3-box diagram in $\mathcal{T}op_*$ of the form:

$$
\begin{array}{ccccccc}
A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \longrightarrow & * \\
\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow \\
 & \longrightarrow & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0
\end{array}
$$

Then, for each set $\{K_3, F_1, F_2, F_3, H_2\}$ of coherent homotopies, the induced diagram

$$
\begin{array}{ccccc}
\Sigma A_3 & \xrightarrow{coext(F_3,K_3)} & \mathcal{M}(f_2,a_2) & \xrightarrow{\mu_{F_2}} & B_1 \\
{\scriptstyle [\Sigma a_3]}\downarrow & {\scriptstyle [q]}\swarrow & \downarrow{\scriptstyle \mu} & {\scriptstyle [j]}\swarrow & \downarrow{\scriptstyle [b_1]} \\
\Sigma A_2 & \xrightarrow[coext(F_2,1_o)]{} & \mathcal{M}(i_1^{b_2}\circ f_1,o) & \xrightarrow[ext(H_2,F_1)]{} & B_0
\end{array}
$$

is a split 2-box diagram in $\mathrm{H}\mathcal{T}op_*$.

Box quaternary bracket operation (small version)

We define the (*small*) *box quaternary bracket operation*

$$\mathcal{Q}_3 \left(\begin{array}{ccccccc} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \longrightarrow & * \\ \downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow \\ * & \longrightarrow & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array} \right) \subset \mathcal{A}_{\mathcal{T}op_*}(o\colon \Sigma A_3 \to B_0) \cong \pi(\Sigma^2 A_3, B_0)$$

to be the union of 2-sided matrix Toda brackets

$$\bigcup \dot{\square} \left(\begin{array}{ccccc} \Sigma A_3 & \xrightarrow{coext(F_3,K_3)} & \mathcal{M}(f_2,a_2) & \xrightarrow{\mu_{F_2}} & B_1 & & \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j} & & \\ & & \Sigma A_2 & \xrightarrow[coext(F_2,1_o)]{} & \mathcal{M}(i_1^{b_2} \circ f_1, o) & \xrightarrow[ext(H_2,F_1)]{} & B_0 \end{array} \right)$$

where the union is taken over all sets of coherent homotopies. It is *defined* (that is, nonvacuous) precisely when a set of coherent homotopies exists as above.

Box quaternary bracket operation (large version)

We define the (*large*) *box quaternary bracket operation*

$$\mathcal{Q}_3^c \begin{pmatrix} A_3 & \xrightarrow{a_3} & A_2 & \xrightarrow{a_2} & A_1 & \longrightarrow & * \\ \downarrow & & f_2\downarrow & & \downarrow f_1 & & \downarrow \\ * & \longrightarrow & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{pmatrix} \subset \mathcal{A}_{\mathcal{T}op_*}(o\colon \Sigma A_3 \to B_0) \cong \pi(\Sigma^2 A_3, B_0)$$

to be the union of box brackets

$$\bigcup \mathcal{L}_2 \begin{pmatrix} \Sigma A_3 & \xrightarrow{coext(F_3,K_3)} & \mathcal{M}(f_2,a_2) & \xrightarrow{\mu_{F_2}} & B_1 \\ [\Sigma a_3]\downarrow & & \mu\downarrow & & \downarrow [b_1] \\ \Sigma A_2 & \xrightarrow[coext(F_2,1_o)]{} & \mathcal{M}(i_1^{b_2}\circ f_1,o) & \xrightarrow[ext(H_2,F_1)]{} & B_0 \end{pmatrix}$$

where the union is taken over all sets of coherent homotopies.

- We note that since $\mathcal{A}_{\mathcal{T}op_*}(o\colon \Sigma A_3 \to B_0) \cong \pi(\Sigma^2 A_3, B_0)$ is an abelian group, the large box quaternary bracket operation $\mathcal{Q}_3^c$ also can be defined as the union of the operations

$$\gamma \begin{pmatrix} \Sigma A_3 & \xrightarrow{coext(F_3,K_3)} & \mathcal{M}(f_2,a_2) & \xrightarrow{\mu_{F_2}} & B_1 & \xrightarrow{b_1} & B_0 \\ \downarrow{id} & & \downarrow{q} & & \downarrow{j} & & \downarrow{id} \\ \Sigma A_3 & \xrightarrow[\Sigma a_3]{} & \Sigma A_2 & \xrightarrow[coext(F_2,1_o)]{} & \mathcal{M}(i_1^{b_2}\circ f_1,o) & \xrightarrow[ext(H_2,F_1)]{} & B_0 \end{pmatrix}$$

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- The inclusion $\mathcal{Q}_3 \subset \mathcal{Q}_3^c$ is valid so that $\mathcal{Q}_3$ possibly has smaller indeterminacy that $\mathcal{Q}_3^c$.

The quaternary Toda bracket

- $$A_4 \xrightarrow{a_4} A_3 \xrightarrow{a_3} A_2 \xrightarrow{a_2} A_1 \xrightarrow{a_1} A_0$$

  Assume homotopies $H_i \colon o \Rightarrow a_{i-1} \circ a_i$, $i = 2, 3, 4$ exist satisfying the coherence conditions:

  $$-a_2 H_4 + H_3 a_4 \sim 1_o \quad (\text{implies } 0 \in \{a_2, a_3, a_4\})$$
  $$-a_1 H_3 + H_2 a_3 \sim 1_o \quad (\text{implies } 0 \in \{a_1, a_2, a_3\})$$

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- Then the 3-box diagram

$$\left(\begin{array}{cccc} A_4 & \xrightarrow{a_4} A_3 & \xrightarrow{a_3} A_2 & \longrightarrow * \\ \downarrow & \downarrow & \downarrow a_2 & \downarrow \\ * & \longrightarrow * & \longrightarrow A_1 & \xrightarrow[a_1]{} A_0 \end{array}\right)$$

admits $\{H_4, H_3, H_2\}$ as a set of coherent homotopies.

- We set

$$\{a_1, a_2, a_3, a_4\}' := Q_3 \subset \pi(\Sigma^2 A_4, A_0)$$

obtaining the quaternary Toda bracket (as considered by Hardie-Kamps-Marcum-Oda (2004)) while

$$\{a_1, a_2, a_3, a_4\} := Q_3^c \subset \pi(\Sigma^2 A_4, A_0)$$

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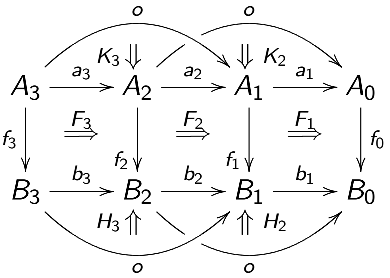
- Of course $\{a_1, a_2, a_3, a_4\}' \subset \{a_1, a_2, a_3, a_4\}$.

The triple box bracket (Marcum–Oda, 2008)



$\rightsquigarrow \quad \mathcal{L}_3 \subset \pi(\Sigma^2 A_3, B_0)$

$\mathcal{L}_3$ is defined if the homotopies satisfy coherence conditions

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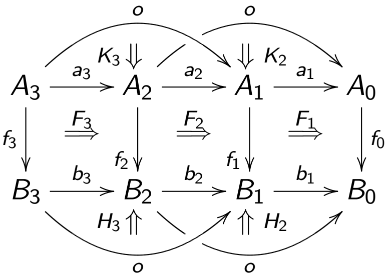

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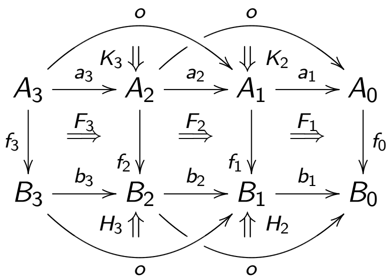

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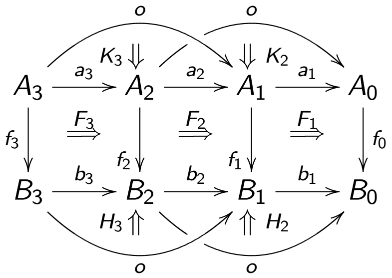

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$\mathcal{L}_3$ is defined if the homotopies satisfy coherence conditions

- Inductively this leads to *long box brackets*
- Can be used to complete computations of some unstable groups
- For example useful in the 25-stem for describing generators of the group $\pi^7_{32} = \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^3$

The box quartet operation (Marcum–Oda, 2009)

$$
\begin{array}{ccccccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B & \xrightarrow{r} & R & \xrightarrow{z} & Z \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle k} \\
U & \xrightarrow[u]{} & A & \xrightarrow[a]{} & X & \xrightarrow[s]{} & Y & \xrightarrow[v]{} & V
\end{array}
\quad \rightsquigarrow \quad \mathcal{D} \subset (\Sigma^2 W, V)
$$

$\mathcal{D}$ is defined if the three associated box brackets are coherent

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- This operation yields new “relations of elements”

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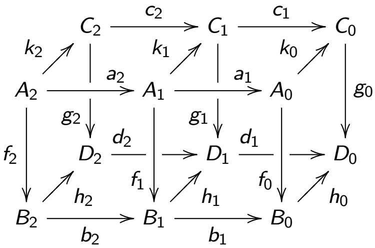
$\mathcal{D}$ is defined if the three associated box brackets are coherent

- This operation yields new “relations of elements”
- For example, in the 23-stem $\bar{\varepsilon}_6 \circ \varepsilon_{21} = 4\nu_6 \circ \bar{\kappa}_9$ in $\pi_{29} S^6$

Double box bracket operation

- Suppose that the 2-cube diagram in $\mathcal{T}op_*$



admits a set of coherent homotopies:

$$\mathcal{U}_2 = \{F_2, F_1, H_2, K_2, G_2, G_1, J_2, L_2, T_2, T_1, S_2, S_1, M_2, M_1, M_0\}$$

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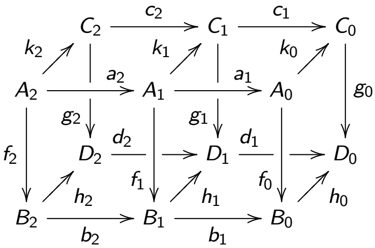

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- Then a 2-box diagram in the homotopy category is obtained:

$$\begin{array}{ccccc}
\Sigma A_2 & \xrightarrow{coext(F_2,K_2)} & \mathcal{M}(i_1^{b_2}\circ f_1, a_1) & \xrightarrow{ext(H_2,F_1)} & B_0 \\
\downarrow{\scriptstyle [\Sigma k_2]} & & \downarrow{\scriptstyle \mu(S_2,M_1,T_1)} & & \downarrow{\scriptstyle [h_0]} \\
\Sigma C_2 & \xrightarrow{coext(G_2,J_2)} & \mathcal{M}(i_1^{d_2}\circ g_1, c_1) & \xrightarrow{ext(L_2,G_1)} & D_0
\end{array}$$

- The operations

$$\square\left(\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 & & \\ & & {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_0} & & \\ & & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right) \text{ and } \mathcal{L}_2\left(\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right)$$

are defined.

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$$\dot{\square}\left(\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ & & {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_0} \\ & & B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right) \text{ and } \mathcal{L}_2\left(\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right) \text{ are}$$

defined.

- Moreover $\mathcal{L}_2\left(\begin{array}{ccccc} A_2 & \xrightarrow{a_2} & A_1 & \xrightarrow{a_1} & A_0 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_2 & \xrightarrow[b_2]{} & B_1 & \xrightarrow[b_1]{} & B_0 \end{array}\right) =$

$$\{f_0, a_1, a_2\} - \{b_1, f_1, a_2\} + \{b_1, b_2, f_2\}$$