Toward Descent Cohomology and Twisted Forms in Homotopy Theory

Jonathan Beardsley

Johns Hopkins University

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Suppose we have a cover $\{U_i \to X\}$ in some topology $\mathcal{T}$ on a category $\mathcal{C}$, and a (pseudo)functor $F : \mathcal{C} \to Cat$.

- A *descent datum* is an element $s_i \in F(U_i)$ for each i, with "gluing data" on intersections $U_i \times_X U_j$ which satisfies certain diagrammatic conditions.
- $\{U_i \to X\}$ *is of effective descent for* F if such a datum uniquely identifies an element of $F(X)$.

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We can rephrase this by saying that $F(X)$ is equivalent to the limit (in Cat) of the following diagram. Recall that that limit is the category of *descent data* for the cover $\{U_i\}$ and the functor F.

$$F(\textstyle\coprod_i U_i) \rightrightarrows F(\coprod_{i,j} U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} F(\coprod_{i,j,k} U_{ijk})$$

If our spaces $\coprod U_i$ and X are just affine schemes, we can think in terms of rings instead. That is, given a morphism of rings in some topology $\mathcal{T}$, $\phi : R \to S$ and a stack F (e.g. the one that to R associates the category of R-modules), the category of descent data is the limit of the diagram:

$$SMod \rightrightarrows (S \otimes_R S)Mod \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} (S \otimes_R S \otimes_R S)Mod$$

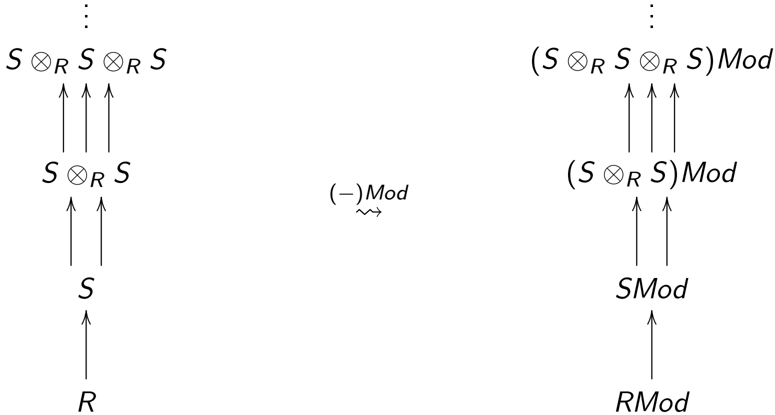
Definition

A descent datum for a morphism of commutative rings $\phi : R \to S$ consists of:

- *an S-module M*
- *an isomorphism (of $S \otimes_R S$-modules), $\theta : p_0^*(M) \xrightarrow{\cong} p_1^*(M)$*
- *a commutative diagram which constitutes the cocycle condition:*

$$
\begin{array}{ccc}
 & p_{10}^*(M) \cong p_{02}^*(M) & \\
{}^{p_0^*(\theta)} \nearrow & & \searrow {}^{p_2^*(\theta)} \\
p_{00}^*(M) \cong p_{01}^*(M) & \xrightarrow{p_1^*(\theta)} & p_{11}^*(M) \cong p_{12}^*(M)
\end{array}
$$

Recall that that limit diagram is actually just $(-)Mod$ applied to the first three levels of the *Amitsur complex* (which we will denote $S/R^{\bullet}$)!



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Definition

Let $\phi : R \to S$ be a faithfully flat morphism of commutative rings, N an R-module, and $N \otimes_R S$ the canonical descent datum generated by N. Then a twisted form for N along ϕ is an R-module N' such that $N' \otimes_R S \cong N \otimes_R S$.

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Theorem (See e.g. Waterhouse, Menini and Ştefan, many others...)

Under the above assumptions, twisted forms of N are in bijection with the set of isomorphism classes of descent data with underlying S-module $N \otimes_R S$.

Definition

For an R-module N, define $Aut(N) : CRng/R \to Group$ by $Aut(N)(S) = Aut_S(S \otimes_R N)$.

Theorem (Ibid.)

The set of twisted forms for N along $\phi : R \to S$ is in bijection with the first (non-abelian) cohomology of the cosimplical group $Aut(N)(R/S^{\bullet})$:

$$Aut(N)(S) \rightrightarrows Aut(N)(S \otimes_R S) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} Aut(N)(S \otimes_R S \otimes_R S) \cdots$$

Recall that in the case that $\phi : R \to S$ is a Galois extension or Hopf-Galois extension of rings, the above cohomology can often be computed as a the group cohomology of a Galois group [Serre], or the Hopf cohomology of the associated Hopf algebra [Nuss and Wambst].

Definition (Lurie)

For $\phi : R \to S$, a map of E_∞ rings in symmetric monoidal ∞-category $\mathcal{C}$, the ∞-category of descent data for ϕ is the totalization of the cosimplicial ∞-category (again based on the Amitsur complex):

$$SMod \rightrightarrows (S \otimes_R S)Mod \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; (S \otimes_R S \otimes_R S)Mod \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \cdots$$

But similarly to above, where we identified a descent datum with an isomorphism and a commuting diagram, we can identify and ∞-descent datum with an invertible 1-cell and a sequence of higher homotopy coherence diagrams:

Definition

Under the assumptions given above, a descent datum for $\phi : R \to S$ *is:*

- *an* S*-module* M,
- *an invertible 1-cell* $\theta : p_0^*(M) \to p_1^*(M)$,
- *a 2-cell*

$$
\begin{array}{ccc}
 & p_{10}^*(M) \simeq p_{02}^*(M) & \\
\overset{p_0^*(\theta)}{\nearrow} & \Updownarrow & \overset{p_2^*(\theta)}{\searrow} \\
p_{00}^*(M) \simeq p_{01}^*(M) & \xrightarrow{p_1^*(\theta)} & p_{11}^*(M) \simeq p_{12}^*(M)
\end{array}
$$

- *higher n-cells satisfying higher cycle conditions...*

Theorem (Pre-theorem/Work-in-Progress)

Given a morphism $\phi : R \to S$ of E_∞ ring-spectra which is of effective descent for modules, and an R-module N, the space of descent data for $N \wedge_R S$ is equivalent to the space of twisted forms for N. Moreover, isomorphism classes of twisted forms of N are in bijection with π_1 of the totalization of the cosimplicial space:

$$hAut(N)(S) \rightrightarrows hAut(N)(S \wedge_R S) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} hAut(N)(S \wedge_R S \wedge_R S) \cdots$$

- In analogy with the discrete case, instead of computing $\check{H}^1$ of a cosimplicial group, we now want to compute π_1 of the totalization of a cosimplicial space!
- This yields a Bousfield-Kan spectral sequence which takes homotopy automorphisms of N and checks that they fit into the necessary coherence diagrams all the way to ∞.
- The totalization of this cosimplicial space is the *space of descent data on* $N \wedge_R S$.

Remark

- Applying π_0 to the above construction recovers descent cohomology and twisted forms for discrete covers.
- In some cases, it seems likely that the above construction can be reinterpreted and simplified if $\phi : R \to S$ is a Galois or Hopf-Galois extension. In other words, a descent datum would correspond to an (co)action of a (Hopf-)Galois (algebra) group.
- There is reason to be interested in Galois extensions and Hopf-Galois extensions of ring spectra!

Recall that there are many examples of Hopf-Galois extensions and Galois extensions in homotopy theory:

- $\mathbb{S} \to MU$ and $S \to X(n)$ are Hopf-Galois extensions for Hopf algebras $\Sigma^\infty_+(BU)$ and $\Sigma^\infty_+(\Omega SU(n))$ [Rognes, Roth], as well as the intermediate Hopf-Galois extensions $X(n) \to X(n+1)$ for $\Sigma^\infty_+(\Omega S^{2n+1})$ [B]
- $L_{K(n)}\mathbb{S} \to E_n$ is a $K(n)$-local Galois extension for $\mathbb{G}_n$, the Morava stabilizer group [Rognes].
- Other Thom spectra give examples of Hopf-Galois extensions, like Baker and Richter's $\mathbb{S} \to M\Xi$, with Hopf-Galois object $\Sigma^\infty_+(\Omega\Sigma\mathbb{C}P^\infty)$ [Roth], as well as (when 2 is inverted) the forgetful morphism $MSp \to MU$, by $\Sigma^\infty_+(Sp/U)$ [Baker and Morava].

- For $\mathbb{S} \to MU$, considering twisted forms of MU might lead to a more algebro-geometric understanding (or different proof altogether) of the Nilpotence Theorem of Devinatz, Hopkins and Smith.
- in the case of $L_{K(n)}\mathbb{S} \to E_n$, descent cohomology would seem to classify actions of $\mathbb{G}_n$ on an E_n-module.

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