Topological Hochschild Homology and Koszul Duality

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Why You Care

Topological Hochschild Homology is

- Defined in analogy with Hochschild homology

$$\mathrm{THH}_n(A) = \underbrace{A \wedge \cdots \wedge A}_{n \text{ times}}$$

- Related to K-theory
- Related to Topological Field Theories

Why You Care: K-Theory Edition

K-theory is an invariant of rings , exact categories , Waldhausen categories, Waldhausen ∞-categories

- K-theory of rings contains arithmetic information (class groups, number of real and complex embeddings, special values of ζ-functions)
- K-theory of topological spaces (Waldhausen A-theory) contains information about pseudo-isotopy groups and thus diffeomorphsim groups
- K-theory of ring spectra is related to chromatic phenomena (red shift conjecture)
- K-theory of schemes is related to intersection theory, motivic stuff, etc.

The Problem With K-Theory

The problem is that K-theory is very difficult to compute.
K-theory admits a map to THH, $K \to \mathrm{THH}$ (generalization of a map $K_* \to \mathrm{HH}_*$) The map lifts to an invariant called topological cyclic homology

$$\begin{array}{ccc} & & \mathrm{TC} \\ & \nearrow & \\ K & \longrightarrow & \mathrm{THH} \end{array}$$

And TC is computable (kinda).

Field Theories

What is a field theory?
It's a manifold invariant

Definition (rough)

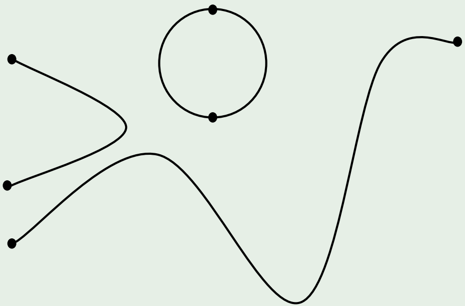
A **topological field theory** is a symmetric monoidal functor from a cobordism category $\mathbf{Cob}^{\amalg}$ to some symmetric monoidal category $\mathcal{C}^{\otimes}$

$$FT : \mathbf{Cob}^{\amalg} \to \mathcal{C}^{\otimes}$$

We'll only care about the oriented cobordism category.

Example

A 0-dimensional example. The following is a cobordism.



We'll pretend the target category is the category of bimodules. That is, the objects are algebras, and the morphisms are bimodules (in spectra).

Example

The bimodule corresponding to a right-facing arc is $A_{A\otimes A^{\mathrm{op}}}$ and the left-facing arc is ${}_{A\otimes A^{\mathrm{op}}}A$. So

$$F(S^1) = A \otimes_{A\otimes A^{\mathrm{op}}} A$$

Not exactly Hochschild homology.
But if we work in homotopical or ∞-categories, we get

$$F(S^1) = A \otimes^{\mathbf{L}}_{A\otimes A^{\mathrm{op}}} A$$

Thus, THH is one of the simplest manifold invariants we get out of a field theory.

(Derived) Koszul Duality

A, B augmented $\mathbf{E}_1$-ring spectra , i.e. $A \to S$, $B \to S$ that give S A and B-module structures.

Definition (Rough)

A and B are **Koszul dual** (roughly) if

$$B \simeq \mathrm{RHom}_A(S, S)$$
$$A \simeq \mathrm{RHom}_B(S, S)$$

Remark

Alternatively, $B \simeq F(S \wedge_A^{\mathbf{L}} S, S)$

Example

X a compact, simply-connected space. Then $\Sigma_+^\infty \Omega X$ and DX are Koszul dual.

Motivation

Example

Let X finite simply connected space. Let $\mathcal{L}X$ denote the free loop space $\mathrm{Map}(S^1, X)$. Then

$$\mathrm{THH}(\Sigma^\infty_+ \Omega X) \simeq \Sigma^\infty_+ \mathcal{L}X$$
$$D(\mathrm{THH}(DX)) \simeq \Sigma^\infty_+ \mathcal{L}X$$

Example

Chain version of this was known (Jones-McCleary)

$$\mathrm{HH}_*(C_*(\Omega X)) \simeq \mathrm{Hom}(\mathrm{HH}_*(C^*(X)), k)$$

Question: Is there a reason for this? Does this hold more generally for Koszul dual algebras?

Theorem (Campbell)

Let A and B be Koszul dual with A compact as an S-module. Then

$$D(\mathrm{THH}(A)) \simeq \mathrm{THH}(B).$$

Corollary

X simply connected, compact, then

$$\mathrm{THH}(\Sigma^\infty_+ \Omega X) \simeq D(\mathrm{THH}(DX))$$

Consequences

Theorem

R any ring spectra. A, B Koszul dual over R. Then

$$D_R(\mathrm{THH}_R(A)) \simeq \mathrm{THH}_R(B).$$

Corollary (Jones-McCleary)

$\mathrm{HH}_*(C_*(\Omega X)) \cong \mathrm{HH}_*(C^*(X))$

Can use previously computed cases of Koszul duality (see e.g. Baker and Lazarev)

Corollary

$\mathrm{MU}_p^\wedge$ *has a Koszul dual over* $H\mathbf{F}_p$, *so*

$$D(\mathrm{THH}(B_p)) \simeq \mathrm{THH}(\mathrm{MU}_p^\wedge)$$

Consequences

Assuming cobordism hypothesis

Theorem (Restatement)

There is a duality in 0-dimensional topological field theories, given by Koszul duality.

Remark

But the duality is not symmetric!

Conjecture

A an $\mathbf{E}_n$-algebra and B its Koszul dual.Let M be an n-manifold.Then

$$D\left(\int_M A\right) = \int_M B$$

Theorem above is

$$D\left(\int_{S^1} A\right) \simeq \int_{S^1} B$$