Infinity Prop(erad)s

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- A **prop(erad)** is a generalization of an ordinary category in which composition is strictly associative.
- In an ordinary category, a morphism $x \xrightarrow{f} y$ has one input and one output.
- We extend the notion of a category by allowing morphisms with **finite** lists of objects as inputs and outputs.

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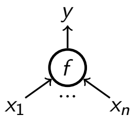
$$(x_1, \ldots, x_n) \xrightarrow{f} y$$

with $n \geq 0$. We often call such a morphism an **operation** and denote it by the following decorated graph.

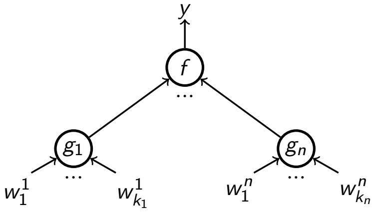


Operads

Composition of operations

$$\left(w_1^i, \ldots, w_{k_i}^i\right) \xrightarrow{g_i} x_i$$

for each i, then the operadic composition $\gamma(f; g_1, \ldots, g_n)$, is represented by the following decorated 2-level tree.



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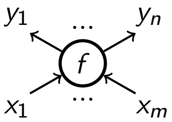
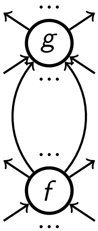
$$(x_1, \ldots, x_m) \xrightarrow{f} (y_1, \ldots, y_n)$$

with $m, n \geq 0$. These operations are visualized as decorated corolla.



Properads

The properadic composition is represented by **partially grafted corollas** like



- This properadic composition is defined when a non-empty sub-list of the outputs of f match a non-empty sub-list of the inputs of g.

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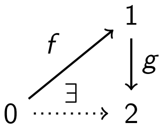

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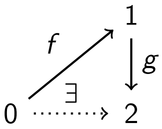

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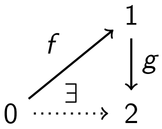

- Any two compositions are homotopic.
- Associativity holds up to homotopy.

To make these ideas precise, we the language of $Set^{\Delta^{op}}$.

- f and g are two 1-simplices in X that determine a unique inner horn $\Lambda^1[2] \to X$, with g as the 0-face and f as the 2-face.

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Definition (Joyal, Lurie, Boardman-Vogt,...)

An ∞-category is an object in $Set^{\Delta^{op}}$ in which every inner horn

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\forall} & X \\ \downarrow & \overset{\exists}{\nearrow} & \\ \Delta[n] & & \end{array}$$

with $0 < k < n$ has a filler.

- To define ∞-properads, we generalize $Set^{\Delta^{op}}$.

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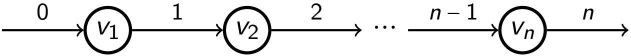
- Notice that Δ can be represented using linear graphs, i.e. the object

$$[n] = \{0 < 1 < \cdots < n\} \in \Delta$$

is the category generated by the linear graph



with n vertices.

- Here each vertex v_i is the generating morphism $i-1 \to i$.

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Definition (Moerdijk-Weiss)

An ∞-operad is an object in $Set^{\Omega^{op}}$ that satisfies an inner horn extension property.

- Properadic compositions and their axioms are parametrized by connected graphs without directed cycles, which we call **connected wheel-free graphs**.

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- Fact: A connected wheel-free graph freely generates a properad, called a **graphical properad**.
- Graphical properads form a (non-full) subcategory Γ of properads. We call Γ the **graphical category**.

Definition (Hackney-R-Yau)

An ∞-properad is an object in $Set^{\Gamma^{op}}$ that satisfies an inner horn extension property.

Connected Wheel-Free Graphs

Fix an infinite set $\mathfrak{F}$.
A **generalized graph** G is a finite set $Flag(G) \subset \mathfrak{F}$ with

- a partition $Flag(G) = \coprod_{\alpha \in A} F_\alpha$ with A finite,
- a distinguished partition subset F_ϵ called the **exceptional cell**
- an involution ι satisfying $\iota F_\epsilon \subseteq F_\epsilon$, and
- a free involution π on the set of ι-fixed points in F_ϵ.

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- Call G an **ordinary graph** if its exceptional cell is empty.

Connected Wheel-Free Graphs

- The elements in $Flag(G)$ are called **flags**.
- Flags in a non-exceptional cell are called **ordinary flags**. Flags in the exceptional cell F_ϵ are called **exceptional flags**.
- Each non-exceptional partition subset $F_\alpha \neq F_\epsilon$ is a **vertex**.
- An empty vertex is an **isolated vertex**
- A flag in a vertex is said to be **adjacent to** or **attached to** that vertex.

Connected Wheel-Free Graphs

- An ι-fixed point is a **leg** of G. An **ordinary leg** (resp., **exceptional leg**) is an ordinary (resp., exceptional) flag that is also a leg.
- An ι-fixed point $x \in F_\epsilon$, the pair $\{x, \pi x\}$ is an **exceptional edge**.
- A 2-cycle of the involution ι consisting of ordinary flags is an **ordinary edge**. A 2-cycle of ι contained in a vertex is a **loop** at that vertex. A 2-cycle of ι in the exceptional cell is an **exceptional loop**.
- An **internal edge** is a 2-cycle of ι, i.e., either an ordinary edge or an exceptional loop.
- An ordinary edge $e = \{e_{-1}, e_1\}$ is said to be **adjacent to** or **attached to** a vertex v if either (or both) $e_i \in v$.

Connected Wheel-Free Graphs

- A **coloring** of G is a function

$$Flag(G) \xrightarrow{\kappa} \mathfrak{C}$$

that is constant on orbits of both involutions ι and π.

- A **direction** of G is a function

$$Flag(G) \xrightarrow{\delta} \{-1, 1\}$$

such that

  - if $\iota x \neq x$, then $\delta(\iota x) = -\delta(x)$, and
  - if $x \in F_\epsilon$, then $\delta(\pi x) = -\delta(x)$.

- For G with direction, an **input** (resp., **output**) of a vertex is a flag x such that $\delta(x) = 1$ (resp., $\delta(x) = -1$).
- An **input** (resp., **output**) of G is a leg x such that $\delta(x) = 1$ (resp., $\delta(x) = -1$).
- A **listing** for G with direction is a choice for each of a bijection of pairs of sets

$$(in(u), out(u)) \xrightarrow{\ell_u} (\{1, \ldots, |in(u)|\}, \{1, \ldots, |out(u)|\}),$$

for each vertex in G

Connected Wheel-Free Graphs

Definition

A $\mathfrak{C}$-**colored wheeled graph**, or just a **wheeled graph**, is a generalized graph together with a choice of a coloring, a direction, and a listing.

Example

The **empty graph** $\varnothing$ has

$$Flag(\varnothing) = \varnothing = \coprod \varnothing,$$

whose exceptional cell is $\varnothing$, and it has no non-exceptional partition subsets. In particular, it has no vertices and no flags.

Example

Suppose n is a positive integer. The **union of n isolated vertices** is the graph V_n with

$$(V_n) = \varnothing = \coprod_{i=1}^{n+1} \varnothing.$$

It has an empty set of flags, an empty exceptional cell, and n empty non-exceptional partition subsets, each of which is an isolated vertex. For example, we can represent V_3 pictorially as

$$\bullet \qquad \bullet \qquad \bullet$$

with each $\bullet$ representing an isolated vertex.

Example

Pick a color $c \in \mathfrak{C}$. The c-**colored exceptional edge** is the graph G whose only partition subset is the exceptional cell

$$Flag(G) = F_\epsilon = \{f_1, f_{-1}\},$$

with

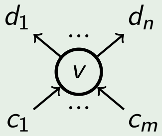
$$\iota(f_i) = f_i, \quad \kappa(f_i) = c, \delta(f_i) = i.$$

It can be represented pictorially as

in which the top (resp., bottom) half is f_{-1} (resp., f_1). Note that this graph has no vertices and has one exceptional edge.

Example

The $(\underline{d};\underline{c})$-corolla can be represented pictorially as the following graph.



$C_{(\underline{d};\underline{c})}$ corolla is the $(\underline{d};\underline{c})$-wheeled graph with :

- $$Flag\left(C_{(\underline{d};\underline{c})}\right) = \{i_1, \ldots, i_m, o_1, \ldots, o_n\}.$$
- $v = Flag(G)$ as its only vertex; its exceptional cell is empty.
- The structure maps: $\iota(i_k) = i_k$ and $\iota(o_j) = o_j$ for all k and j
- $\kappa(i_k) = c_k$ and $\kappa(o_j) = d_j$.
- $\delta(i_k) = 1$ and $\delta(o_j) = -1$, and $\ell_u(i_k) = k$ and $\ell_u(o_j) = o_j$ for $u \in \{C_{(\underline{d};\underline{c})}, v\}$.

Connected Wheel-Free Graphs

- A **path** in G is a pair

$$P = \left(\left(e^j\right)_{j=1}^{r}, \left(v_i\right)_{i=0}^{r} \right)$$

with $r \geq 0$, in which

  - the v_i are distinct vertices except possibly for $v_0 = v_r$,
  - the e^j are distinct ordinary edges, and
  - each e^j is adjacent to both v_{j-1} and v_j.

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- A **directed path** in G is an internal path P as above such that each e^j has initial vertex v_{j-1} and terminal vertex v_j.

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- A **directed path** in G is an internal path P as above such that each e^j has initial vertex v_{j-1} and terminal vertex v_j.
- A **wheel** in G is a directed path that is also a cycle.

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2. G is a single exceptional loop.
3. G satisfies all of the following conditions.
   - G is ordinary (i.e., has no exceptional flags).
   - G is *not* the empty graph.
   - For any two distinct vertices u and v in G, there exists an internal path in G with u as its initial vertex and v as its terminal vertex.

Definition

Let Γ denote the (not full) subcategory of properads generated by connected wheel-free graphs.

Definition

An ∞-properad is an object in $Set^{\Gamma^{op}}$ that satisfies an inner horn extension property.

Final Notes

- Whereas every object in the finite ordinal category Δ and the dendroidal category Ω has a finite set of elements, most objects in the graphical category Γ have infinite sets of elements.
- the graphical analogs of the cosimplicial identities are not entirely straightfoward to prove, i.e.

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- the graphical analogs of the cosimplicial identities are not entirely straightfoward to prove, i.e. HARD.
- General properad maps between them may exhibit bad behavior that would never happen in Δ and Ω.

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- $Set^{\Gamma^{op}}$ admits a cofibrantly generated model category structure (in progress).
- Can be used to study bi-algebra structures the way that ∞-operads are used to study algebra structures.