Gorenstein homological algebra

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Injectivity and finiteness

Homological algebra is, by and large, the study of approximating modules by projective, flat, or injective modules. I want to convince you that the standard notions of flat and injective modules depend on a notion of finiteness, and this notion of finiteness is not always the appropriate one, and so our notions of flat and injective modules are not always the appropriate ones.

As you know, N is an injective (left) R-module if and only if

$$\operatorname{Ext}^1_R(M, N) = 0 \text{ for all } R\text{-modules } M.$$

But Baer's criterion tells us that in fact N is injective if and

$$\operatorname{Ext}^1_R(R/I, N) = 0 \text{ for all left ideals } I$$

or, put more invariantly, N is injective if and only if

$$\operatorname{Ext}^1_R(M, N) = 0 \text{ for all finitely generated } M.$$

However, it is well-known that finitely generated left R-modules are well-behaved if and only if R is left Noetherian.

The Noetherian case

More precisely, we have

Proposition

The collection of finitely generated left R-modules is a thick subcategory if and only if R is left Noetherian.

Here a collection of modules is **thick** if and only if it is closed under summands and whenever we have a short exact sequences with 2 out of 3 entries in the collection, so is the third.
So there are many propositions about injective modules that are only true over left Noetherian rings, such as direct sums of injectives being injective.
So if we want to do homological algebra over non-Noetherian rings, maybe it behooves us to change the definition of injective a little. If finitely generated modules are not good enough, how about finitely presented modules?

The coherent case

Definition

A module N is called **absolutely pure** or **FP-injective** if $\mathrm{Ext}^1_R(M, N) = 0$ for all finitely presented R-modules M.

However, we have a similar problem.

Proposition

The collection of finitely presented R-modules is a thick subcategory if and only if R is left coherent (that is, if every finitely generated left ideal is finitely presented).

Because of this, absolutely pure modules are better behaved over left coherent rings than in general. For example, direct limits of absolutely pure modules are absolutely pure if and only if R is left coherent. Of course, absolutely pure coincides with injective if R is left Noetherian.

The general case–FP_∞-modules

By and large, workers in commutative algebra and algebraic geometry deal with this by assuming that all rings are coherent. This is a problem in stable homotopy theory, because the homotopy of the sphere is definitely not coherent.

So what do we need to do in general? Well, finitely presented means that both the generators and the relations are finitely generated. We should assume that the relations between the relations between the relations. . . are also finitely generated.

Definition

An R-module is said to have **type** FP_∞ if it has a resolution by finitely generated projectives.

These are used in geometric group theory–Bieri and Kropholler, in particular, have looked at them, as has Benson. Note that R is left coherent precisely when every finitely presented module is FP_∞, and there can be situations when there are hardly any modules of type FP_∞.

Absolutely clean modules

For example, if $R = k[x_1, x_2, \dots]/m^2$, where $m = (x_1, x_2, \dots)$, the only modules of type FP_∞ are finitely generated free. In particular, every module is a direct limit of FP_∞-modules if and only if R is left coherent.

Bieri proved that

Proposition

The collection of modules of type FP_∞ is always a thick subcategory.

So then we make the following definition.

Definition

A module N is **absolutely clean** if $\operatorname{Ext}^1_R(M, N) = 0$ for FP_∞ M.

The absolutely clean modules are then always closed under direct limits, and there is a set of them which generates all the rest under transfinite composition.

Level modules

A somewhat similar story can be told for flat modules. In this case a left module N is flat if and only if

$$\mathrm{Tor}_1^R(M, N) = 0 \text{ for all right } R\text{-modules } M,$$

or, because tensor commutes with direct limits,

$$\mathrm{Tor}_1^R(M, N) = 0 \text{ for all finitely presented right } R\text{-modules } M.$$

So we should expect flat left R-modules to be well-behaved if and only if R is **right** coherent. So, for example, products of flat left R-modules are flat if and only if R is right coherent.

Definition

A left R-module N is **level** if $\mathrm{Tor}_1^R(M, N) = 0$ for all right FP_∞ M.

So level modules coincide with flat modules if and only if R is right coherent, and products of level modules are always level.

Duality

There is a partial duality between flat and injective modules, that we can now understand better. Recall that the **character module** M^+ of a left R-module M is the right R-module

$$M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/Z).$$

This should look familiar; if IX denotes the Brown-Comenetz dual of a spectrum X, then $\pi_*(IX) = (\pi_* X)^+$.
The partial duality goes as follows:

Theorem

An R-module M is flat if and only if M^+ is injective. If R is left Noetherian, then M is injective if and only if M^+ is flat.

We can now fix this partial duality:

Theorem

An R-module M is level if and only if M^+ is absolutely clean, and M is absolutely clean if and only if M^+ is level.

The stable module category

Now let us try to do some homological algebra. Consider the **stable module category**. This is usually applied when the ring R is **quasi-Frobenius**, which means that projective and injective modules coincide. Recall that this happens in particular for $R = k[G]$, where k is a field and G is a finite group. If the characteristic of the field does not divide the order of the group, then $k[G]$ is semisimple, which means that every module is both projective and injective, and we get classical representation theory, where every representation is the sum of irreducible representations. If the characteristic of the field does divide $|G|$, on the other hand, $k[G]$ is quasi-Frobenius but not usually semisimple, and the resulting modular representation theory is more subtle. In this case, one usually forms the stable module category $\mathrm{Stmod}(R)$ by letting

$$\mathrm{Stmod}(R)(M, N) = \mathrm{Hom}_R(M, N)/\sim$$

where $f \sim g$ if $g - f$ factors through a projective module.

Properties of the stable module category

The stable module category category is no longer abelian, but it is triangulatedwhen R is quasi-Frobenius. The suspension of M is the first cosyzygy J/M where J is an injective module containing M, and desuspension is the first syzygy of M.
So we end up with a functor

$$R\text{-Mod} \to \text{Stmod}(R)$$

from an abelian category to a triangulated category that is exact, sends projectives and injectives to 0, and is minimal with respect to these properties.
The question that began this project was: can we do this for an arbitrary ring R?

Abelian model categories

Our approach to this problem is through model categories, as a way of contructing triangulated categories. So what we want is a model structure on the category of R-modules. Since we want the functor from R-modules to the homotopy category of this model structure to be exact, it seems like we should make the model structure **abelian**; this means that cofibrations are monomorphisms, fibrations are epimorphisms, a map is a cofibration if and only if it is a monomorphism with a cofibrant cokernel, and a map is a fibration if and only if it is an epimorphism with fibrant kernel.
I introduced abelian model categories some time ago, and proved that they are determined by classes of objects instead of classes of maps. That is, an abelian model structure is equivalent to three classes of objects $\mathcal{C}$ (the cofibrant objects), $\mathcal{F}$ (the fibrant objects), and $\mathcal{W}$ (the trivial objects), such that $\mathcal{W}$ is thick and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are **complete cotorsion pairs**.

Cotorsion pairs

Here $(\mathcal{D}, \mathcal{E})$ is a complete cotorsion pair if

1. $D \in \mathcal{D}$ if and only if $\text{Ext}^1(D, E) = 0$ for all $E \in \mathcal{E}$ (this is part of lifting)
2. $E \in \mathcal{E}$ if and only if $\text{Ext}^1(D, E) = 0$ for all $D \in \mathcal{D}$.
3. For every M, there is a short exact sequence
$$0 \to E \to D \to M \to 0$$
with $D \in \mathcal{D}$ and $E \in \mathcal{E}$ (this is half of factorization)
4. For every M there is a short exact sequence
$$0 \to M \to E' \to D' \to 0$$
with $E' \in \mathcal{E}$ and $D' \in \mathcal{D}$.

For example, when R is quasi-Frobenius, we can take $\mathcal{C}$ and $\mathcal{F}$ to be everything, and $\mathcal{W}$ to be the projective=injective modules. In this case, the two complete cotorsion pairs are (projective, everything) and (everything, injective). Another example of a cotorsion pair is (flat, cotorsion).

The stable module category for Gorenstein rings

Note that, in any abelian model structure, the projective modules will be in $\mathcal{C} \cap \mathcal{W}$ and the injective modules will be in $\mathcal{F} \cap \mathcal{W}$. Since $\mathcal{W}$ is thick, it must also contain all modules of finite projective or finite injective dimension. A ring is called **Iwanaga-Gorenstein** if it is left and right Noetherian and (left or right) modules of finite projective dimension coincide with modules of finite injective dimension.

Theorem (H., 2002)

If R is Iwanaga-Gorenstein, there are two Quillen equivalent abelian model structures on R-Mod with $\mathcal{W}$ being the modules of finite projective dimension. In one model structure, everything is cofibrant and $\mathcal{F}$ is the N for which $\operatorname{Ext}^1_R(M, N) = 0$ for all injective M. In the other, everything is fibrant and $\mathcal{C}$ is the M for which $\operatorname{Ext}^1_R(M, N) = 0$ for all projective N.

The homotopy category is a stable module category.

The general case with everything cofibrant

In general, we need a larger $\mathcal{W}$. Let's try with $\mathcal{C}$ being everything. Then $\mathcal{W} = \mathcal{C} \cap \mathcal{W}$ will contain the injectives. But the left half of a cotorsion pair is always closed under direct sums and, more generally, transfinite extensions. Here M is a transfinite extension of a class $\mathcal{D}$ if there is an ordinal β and a colimit-preserving functor from β to R-modules such that $M_0 = 0$ and $M_\alpha \to M_{\alpha+1}$ is a monomorphism with cokernel in $\mathcal{D}$. So $\mathcal{W}$ is a class closed under transfinite extensions that contains the injectives, and we really only know one nontrivial example of that in general: the absolutely clean modules. So it is tempting to say that $\mathcal{W}$ should just be the absolutely clean modules. This would be wrong though, because it also contains the projectives.

So we would like to let $\mathcal{F}$ be the collection of all N such that $\operatorname{Ext}^1_R(M, N) = 0$ for all absolutely clean M, and the define $\mathcal{W}$ to be all the M such that $\operatorname{Ext}^1_R(M, N) = 0$. The problem with this plan is that $\mathcal{W}$ will not be thick.

At this point we have

$$\mathcal{F} = \{N | \operatorname{Ext}^1(M, N) = 0 \text{ for all AC } M\}$$

and

$$\mathcal{W} = \{M | \operatorname{Ext}^1(M, N)) = 0 \text{ for all } N \in \mathcal{F}\}.$$

If

$$0 \to A \to B \to C \to 0$$

is exact, and A and C are in $\mathcal{W}$, then B certainly will be. If B and C are in $\mathcal{W}$, then $\operatorname{Ext}^1(A, F)$ will be trapped between $\operatorname{Ext}^1(B, F)$ and $\operatorname{Ext}^2(C, F)$. So the first thing to do is to make sure $\operatorname{Ext}^n_R(M, N) = 0$ for all $M \in \mathcal{W}$ and $N \in \mathcal{F}$ and $n \geq 1$; that is, we want to make the cotorsion pair $(\mathcal{W}, \mathcal{F})$ **hereditary**. This is not too hard; we just redefine $\mathcal{F}$ as the collection of all N such that $\operatorname{Ext}^n_R(M, N) = 0$ for all absolutely clean M and all $n \geq 1$ and this will do the job.

Making $\mathcal{W}$ thick, part two

Recall that we have

$$0 \to A \to B \to C \to 0$$

and we have made $(\mathcal{W}, \mathcal{F})$ hereditary, making $\mathcal{W}$ almost thick. But we will still not be able to conclude that $C \in \mathcal{W}$ if A and B are. We will be able to conclude that $\text{Ext}^n_R(C, F) = 0$ for all $F \in \mathcal{F}$ and $n \geq 2$ though. If let F' be the first cosyzygy of F (that is, the cokernel of a monomorphism from F to an injective), then this means that $\text{Ext}^1_R(C, F') = 0$ for all $F \in \mathcal{F}$. We need to get from here to the claim that $\text{Ext}^1_R(C, F) = 0$ for ALL $F \in \mathcal{F}$. The easiest way for this to happen would be for every F to be an F'. That is, we would like the following to be true: for every $F \in \mathcal{F}$, there is an $\mathcal{F}' \in \mathcal{F}$ and a short exact sequence

$$0 \to F' \to I \to F \to 0$$

in which I is injective. If we knew this, then $\mathcal{W}$ would be thick.

We have decided that, for every $F \in \mathcal{F}$, we want an exact sequence

$$0 \to F' \to I \to F \to 0$$

in which I is injective. By iterating this, we would get a resolution of F by injectives,

$$\cdots I_2 \to I_1 \to I_0 \to F \to 0$$

in which all the kernels are in $\mathcal{F}$. We could then splice this an the usual injective coresolution to get an exact complex of injectives X with $Z_0X = F$ and in fact all $Z_iX \in \mathcal{F}$. (This is true for i negative as well because we have already made $(\mathcal{W}, \mathcal{F})$ hereditary).
But more is true. If we look at $\operatorname{Hom}(M, X)$ where M is absolutely clean, this will still be exact, because $\operatorname{Ext}^1_R(M, Z_iX) = 0$.
Except for replacing injective by absolutely clean, this is the definition of a **Gorenstein injective** module F. So we should take $\mathcal{F}$ to be the class of **Gorenstein AC-injectives**; that is, those modules F for which there exists an exact complex X of injectives with $Z_0X = F$ and $\operatorname{Hom}(M, X)$ exact for all absolutely clean M.

After all this we get our first version of the stable module category.

Theorem

For any ring R, there is a stable abelian cofibrantly generated model structure on R-Mod in which everything is cofibrant and the fibrant objects are the Gorenstein AC-injectives. The functor from R-Mod to the homotopy category of this model structure is an exact functor to a triangulated category that kills all projectives, all injectives, and all absolutely clean modules.

Note that we have to prove this using chain complexes. This is because it is fairly easy to find a set of chain complexes S such that X is an exact complex of injectives with $\operatorname{Hom}(M, X)$ exact for all AC M if and only if $\operatorname{Ext}(A, X) = 0$ for all $A \in S$, where that Ext is in the category of chain complexes.

We also get a similar theorem when we decide to make $\mathcal{F}$ be everything. In this case, $\mathcal{W} = \mathcal{F} \cap \mathcal{W}$ is going to contain the projective modules. It is also going to be closed under products and inverse transfinite extensions. Again, the smallest thing that we know with this property in general is the level modules, but we can't just take $\mathcal{W}$ to be equal to the level modules, because it also contains the injective modules. So we would like to take $\mathcal{C}$ to be the collection of all M such that $\operatorname{Ext}^1_R(M, N) = 0$ for all level N. But $\mathcal{W}$ won't be thick, so similar considerations lead us to defining $\mathcal{C}$ as modules M that can be written as $Z_0 X$ where X is an exact complex of projectives such that $\operatorname{Hom}_R(X, N)$ remains exact for all level N. The duality between absolutely clean and level modules allows us to prove, with some nontrivial work, that this is the same as $K \otimes_R X$ being exact for all absolutely clean right modules K, generalizing a theorem of Murfet and Salarian in the Noetherian case. So we call these modules **Gorenstein AC-projective**.

The projective stable module category

Theorem

For any ring R, there is a stable abelian cofibrantly generated model structure on R-Mod in which everything is fibrant and the cofibrant objects are the Gorenstein AC-projectives. The functor from R-Mod to the homotopy category of this model structure is an exact functor to a triangulated category that kills all projectives, all injectives, and all level modules.

In the case of an Iwanaga-Gorenstein ring, these two homotopy categories coincide and match with the previous one. In general we think they are different. In some sense they are dual, so we expect them to be equivalent when R has a dualizing complex, but we have not proved this yet.

In particular, this construction gives a definition of Tate cohomology (which is just Hom in the stable module category of $k[G]$) for any group G. We do not yet know if this coincides with other generalizations of Tate cohomology.

Some details: cofibrant generation

As always with model categories, we want to use the small object argument and this requires a generating set. So a cotorsion pair $(\mathcal{D}, \mathcal{E})$ is **cogenerated by a set** if there is a set S such that $\mathcal{E}$ is the class of all E such that $\operatorname{Ext}^1_R(D, E) = 0$ for all $D \in \mathcal{S}$. We need this to be true to have any chance.
Here is the key technical lemma for this.

Lemma

If $\mathcal{A}$ is a class of modules that is closed under taking pure submodules and quotients by pure submodules, there is a subSET S of $\mathcal{A}$ such that every element of $\mathcal{A}$ is a transfinite extension of elements of S.

This follows from the fact we learned from Enochs and Jenda that every module has a pure submodule of small size. This applies to flat modules (and was crucial in the proof of the flat cover conjecture), level modules, absolutely pure modules, and absolutely clean modules.

Some details: duality

Here is the proof that if the left module N is absolutely clean then the right module N^+ is level. Let M be an FP_∞ left module. We need to show $\mathrm{Tor}_1^R(N^+, M) = 0$. Take a short exact sequence

$$E\colon\ 0 \to M_1 \to P \to M \to 0$$

where P is finitely generated projective and M_1 is also of type FP_∞, and in particular finitely presented. Since N is absolutely clean, $\mathrm{Hom}_R(E, N)$ is exact, and so $(\mathrm{Hom}_R(E, N))^+$ is also exact. But

$$(\mathrm{Hom}_R(E, N))^+ \cong N^+ \otimes_R E.$$

This is a duality thing we learned from Enochs and Jenda. There is a natural map $\phi\colon\ N^+ \otimes M \to \mathrm{Hom}_R(M, N)^+$ defined by $\phi(f \otimes m)(h) = f(h(m))$. This map is a natural transformation of right exact functors of M that is an isomorphism when $M = R$, so for all finitely presented M. It then follows immediately that $\mathrm{Tor}_1^R(N^+, M) = 0$, so N^+ is level.

Some details: pure exact complexes

Recall that a complex X is pure exact if $\operatorname{Hom}(F, X)$ is exact for all finitely presented F. The following theorem generalizes a result of Neeman.

Theorem

Let R be a ring and let C be a complex of projective R-modules. If Y is a pure exact complex of R-modules, then $\operatorname{Hom}(C, Y)$ is exact, or, equivalently, every chain map from C to such a Y is chain homotopic to 0. Similarly, if Z is a pure exact complex of right R-modules, then $Z \otimes_R C$ is exact.

Here we are taking the chain complex Hom, whose 0-cycles are the chain maps from C to Y. At first glance it might seem like $\operatorname{Hom}(C, Y)$ should be exact for any exact Y, since C is built out of projectives. But you have nowhere to start an induction, and it is easy to construct a counterexample: take $R = \mathbb{Z}/4$ and $C = Y$ the complex that is R in every dimension with the differential being multiplication by 2.

Some details: more pure exact complexes

This theorem gets at the difference between DG-projective or cofibrant chain complexes C, which are complexes of projectives such that $\mathrm{Hom}(C, Y)$ is exact for all exact Y, and general complexes of projectives.
The proof of this theorem is a lot of elaboration of Kaplansky's theorem that every projective is a direct sum of countably generated projectives. More precisely, using Kaplansky's theorem, we show that the cotorsion pair whose left half is complexes of projectives is cogenerated by bounded above complexes of finitely generated free modules. We can then do a complicated induction to show that if Y is pure exact and C is a bounded above complex of finitely generated projectives then every chain map from C to Y is chain homotopic to 0.