Higher Chromatic Analogues of Twisted K-theory

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K	complex K-theory spectrum
$K(\mathbb{Z}, n)$	Eilenberg-Maclane space
$\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$	infinite orthogonal group

There is a map of spectra $\Sigma^\infty \mathbb{C}P^\infty_+ \to K$ that represents the action of line bundles on K-theory. This map gives rise to a $\Sigma^\infty \mathbb{C}P^\infty_+$-module structure on K:

$$\Sigma^\infty \mathbb{C}P^\infty_+ \wedge K \to K \wedge K \to K$$

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Precisely, if L is the universal line bundle on $\mathbb{C}P^\infty$ and ξ_n is the universal bundle on $BU(n)$, the tensor bundle $L \otimes \xi_n$ is an n-dimensional bundle over $\mathbb{C}P^\infty \times BU(n)$ and is classified by a map

$$\mathbb{C}P^\infty \times BU(n) \longrightarrow BU(n)$$

- Let X be a space and $H \in H^3(X, \mathbb{Z})$.
- H is classified by a map $X \to K(\mathbb{Z}, 3)$, and the induced bundle from the path loop fibration on $K(\mathbb{Z}, 3)$ is a principal $\mathbb{C}P^\infty$-bundle $P \to X$ on X.

$$
\begin{array}{ccc}
\mathbb{C}P^\infty & \longrightarrow & \mathbb{C}P^\infty \\
\downarrow & & \downarrow \\
P & \longrightarrow & PK(\mathbb{Z}, 3) \\
\downarrow & & \downarrow \\
X & \xrightarrow{H} & K(\mathbb{Z}, 3)
\end{array}
$$

So, we have "action" maps:

- $\Sigma^\infty \mathbb{C}P^\infty_+ \wedge \Sigma^\infty P_+ \to \Sigma^\infty P_+$
- $\Sigma^\infty \mathbb{C}P^\infty_+ \wedge K \to K$

We form the "Generalized Thom spectrum"

$$\Sigma^\infty P_+ \wedge_{\Sigma^\infty \mathbb{C}P^\infty_+} K$$

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Definition

The H-twisted K-homology of X is

$$K_*(X, H) = \pi_*(\Sigma^\infty P_+ \wedge_{\Sigma^\infty \mathbb{C}P^\infty_+} K)$$

- If $H = 0$, then $K_*(X, H) = K_*X$, as expected.
- Twisted K-theory has application in mathematical physics. Certain invariants take values in twisted K-theory, rather than untwisted K-theory
- Recent works of Freed-Hopkins-Teleman show that twisted equivariant K-theory is related to Verlinde algebra.

How do you compute twisted K-theory?

There is a spectral sequence

$$\mathrm{Tor}_{s,t}^{K_*(\mathbb{C}P^\infty)}(K_*P, K_*) \Rightarrow K_*(X, H)$$

- Here K_* is a $K_*(\mathbb{C}P^\infty)$-module via the ring map

$$K_*(\mathbb{C}P^\infty) \to K_*K \to K_*$$

- K_*P is a $K_*(\mathbb{C}P^\infty)$-module via the action of $\mathbb{C}P^\infty$ on P.

A universal coefficient isomorphim

Theorem (K.)

In the spectral sequence above,

$$Tor_{s,t}^{K_*(\mathbb{C}P^\infty)}(K_*P, K_*) = 0$$

for all $s > 0$ *and so*

$$K_*(X,H) \cong K_*P \otimes_{K_*(\mathbb{C}P^\infty)} K_*$$

Note that:

- The map $K_*(\mathbb{C}P^\infty) \to K_*$ is far from being flat!
- In general K_*P is not flat over $K_*(\mathbb{C}P^\infty)$.

Key steps in the proof

- Not only is K_*P a $K_*(\mathbb{C}P^\infty)$-module, it's a comodule over K_*K, the algebra of cooperations of K-theory. It's enough to show that the functor

$$- \otimes_{K_*(\mathbb{C}P^\infty)} K_*$$

  is exact on the category of $K_*(\mathbb{C}P^\infty)$-modules which are comodule over K_*K.

- The rings $K_*(\mathbb{C}P^\infty)$ and K_*K are very close!

What do I mean by "the rings $K_*(\mathbb{C}P^\infty)$ and K_*K are very close"?

Theorem (Snaith)

Let $\beta : S^2 \to \Sigma^\infty \mathbb{C}P^\infty_+$ be the Bott map. Then the inclusion $\mathbb{C}P^\infty \cong BU(1) \to BU \times \mathbb{Z}$ localizes to an equivalence

$$\Sigma^\infty \mathbb{C}P^\infty_+ [\beta^{-1}] \simeq K$$

Thus

$$K_*(\mathbb{C}P^\infty)[\beta^{-1}] \cong K_*K$$

- The element β maps to 1 under the map $K_*(\mathbb{C}P^\infty) \to K_*$. So,

$$\begin{aligned} K_*P \otimes_{K_*(\mathbb{C}P^\infty)} K_* &\cong K_*P\,[\beta^{-1}] \otimes_{K_*(\mathbb{C}P^\infty)[\beta^{-1}]} K_* \\ &\cong K_*P\,[\beta^{-1}] \otimes_{K_*K} K_* \end{aligned}$$

- It's enough to prove that the functor

$$- \otimes_{K_*K} K_*$$

is exact on the category of K_*K-modules which are comodules over K_*K.

In short (K_*K, K_*) is a Hopf Algebroid, and $K_*P\left[\beta^{-1}\right]$ is a comudule over it.

Quite generally, suppose (S, k) is a commutative Hopf algebroid, and M is a comodule over it. Then

Theorem (K.)

The functor $- \otimes_S k$ is exact from the category of comodules M over S which are also modules over S via a map of comodules $S \otimes_k M \to M$ to the category of groups.

$\mathbb{G}_n$: Extended Morava stabilizer group

$S\mathbb{G}_n$: Kernel of the homomorphism $\det : \mathbb{G}_n \to \mathbb{Z}_p^\times$

E_n : Morava E-theory

Let R_n denote the homotopy fixed point spectrum of E_n under $S\mathbb{G}_n$ action, that is

$$R_n = E_n^{hS\mathbb{G}_n}$$

Theorem (C. Westerland)

There exist a map $\rho_n : S< det > \to L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+$ *and an equivalence of* E_∞*-ring spectra*

$$L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+[\rho_n^{-1}] \to R_n$$

Note: When $n = 1$, R_1 is the p-adic K-theory, and ρ_1 is the $K(1)$-localization of β.

- There exist a map $L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1) \to R_n$.
- For a $K(n)$-local space X equipped with a class $H \in H^{n+2}(X, \mathbb{Z}_p)$, we define the $K(n)$-local twisted R_n-theory of X to be

$$R_{n*}(X, H) = \pi_*(\Sigma^\infty P_+ \wedge_{L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)} R_n)$$

where $P \to X$ is the bundle induced by H.

Theorem (K.)

There is an isomorphism

$$R_{n*}(X, H) \cong R_{n*}P \otimes_{R_{n*}(L_{K(n)}\Sigma^\infty K(\mathbb{Z}_p, n+1)_+)} R_{n*}$$

Note: When $n = 1$, this result is the $K(1)$-local version of the universal coefficient isomorphism for twisted K-theory.