Equivariant Algebraic $K$-theory

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Algebraic $K$-theory functor

\[ \text{Rings} \xrightarrow{K_i} \text{Groups} \]
The input category $\text{Rings}$ can be replaced with $\text{ExactCat}$ or $\text{WaldhausenCat}$. 
Why do we care about algebraic $K$-theory?

Algebraic $K$-theory is the meeting ground for various other subjects such as

- algebraic geometry,
- number theory,
- algebraic topology,

and it encodes deep information about these.
Why do we care about algebraic $K$-theory?

**Example (Number theory)**

Let $\mathbb{Q}(\zeta_p)^+$ be the maximal real subfield of $\mathbb{Q}(\zeta_p)$ and $h_{\mathbb{Q}(\zeta_p)^+}$ its class number. Then

$$p \nmid h_{\mathbb{Q}(\zeta_p)^+} \iff K_{4i}(\mathbb{Z}) = 0 \text{ for all } i.$$  

**Example (Manifold theory)**

The Waldhausen $K$-theory spectrum of a manifold $M$ decomposes as

$$A(M) \simeq \sum_{\infty} M_+ \times WH^{PL}(M),$$

where the second factor encodes information about pseudo-isotopies on $M.$
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Example (Number theory)
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Example (Manifold theory)
The Waldhausen $K$-theory spectrum of a manifold $M$ decomposes as

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where the second factor encodes information about pseudo-isotopies on $M$. 
**G-actions in algebraic K-theory, a motivating example**

Suppose $E/F$ is a (finite) Galois extension with Galois group $G$. Then $G$ acts on $KE$ and

$$(KE)^G = KF.$$

We have a map from the fixed points to the homotopy fixed points

$$(KE)^G \to (KE)^{hG}$$

induced by

$$(KE)^G = \text{Map}(\ast, KE)^G \to \text{Map}(EG, KE)^G.$$
Initial Quillen-Lichtenbaum conjecture

**Conjecture (Lichtenbaum-Quillen)**

The map

\[ KF \rightarrow KE^hG \]

is an equivalence after p-completion.

**Theorem (Thomason)**

The conjecture becomes true only after inverting a “Bott element” and reducing mod a prime power.
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My goal

The goal is to tell an equivariant story...

Question: What kind of equivariant spectra are there?

- **naive** $G$-spectra (spectra with $G$-action)
- **genuine** $G$-spectra (indexed on $G$-representations, can suspend and desuspend with respect to representation spheres $S^V$, give rise to $RO(G)$-graded cohomology theories)

Naive $G$-spectra are the ones that arise naturally when $G$ acts on the input.

But, genuine $G$-spectra are the “right” objects for equivariant stable homotopy theory.
My goal

GOAL: Encode a "naive" action on the input ring (or category) as a genuine $G$-spectrum.

\[
\begin{array}{ccc}
G\text{-Rings} & \xrightarrow{K_G} & \text{Genuine } G\text{-Spectra} \\
\downarrow & & \downarrow \pi_i \\
& & \text{Mackey functors}
\end{array}
\]
Equivariant algebraic $K$-theory functor

Theorem (M.)

There is a functor $K_G : G\text{-rings} \rightarrow$ genuine $G$-spectra, which applied to

- the topological ring $\mathbb{C}$ with trivial $G$-action $\rightarrow$ yields $KU_G$
- the topological ring $\mathbb{R}$ with trivial $G$-action $\rightarrow$ yields $KO_G$
- the topological ring $\mathbb{C}$ with $\mathbb{Z}/2$ conjugation action $\rightarrow$ yields Atiyah’s $KR$
Theorem, continued

We recover the original Quillen-Lichtenbaum conjecture

**Theorem (M.)**

Let $E/F$ be a Galois extension with group $G$. The map of spectra

$$K_G(E)^G \rightarrow K_G(E)^{hG}$$

from fixed points to homotopy fixed points of the genuine $G$-spectrum $K_G(E)$ is equivalent to the map

$$K(F) \rightarrow K(E)^{hG},$$

where $K(E)^{hG}$ denotes the homotopy fixed points of the naive spectrum $K(E)$.

In particular, $K_G(E)^G \simeq K(\mathbb{Q})$ for any Galois extension $E/\mathbb{Q}$!
Construction of the functor $K^G$

2 steps:

1. good space level definition
2. equivariant delooping of this space
Step 1

Recall (or take as definition):

\[ BGL(R)^+ \simeq \text{group completion of } \coprod_n BGL_n(R) \text{ .} \]

Idea: Replace by equivariant \((G, GL(R) \rtimes G)\)-bundles when \(G\) acts on \(R\).
Equivariant bundle theory

Let \( C \) be a \( G \)-category. Define:

\[
\tilde{G} := \text{translation category of } G
\]

\[
\text{Cat}(\tilde{G}, C) := \text{functor category, with } G \text{ acting by conjugation}
\]

NOTES: \( \text{Cat}(\tilde{G}, C) \cong C \) nonequivariantly, and \( B\tilde{G} \cong EG \).

Theorem (Guillou, May, M.)

Suppose \( G \) acts on \( \Pi \). The canonical map

\[
B\text{Cat}(\tilde{G}, \tilde{\Pi}) \rightarrow B\text{Cat}(\tilde{G}, \Pi),
\]

is a universal principal \((G, \Pi \rtimes G)\)-bundle.
Definition of $K$-theory space, equivariant plus-construction

Suppose $R$ is a $G$-ring.

**Definition**
Define $K_G(R)$ to be the equivariant group completion of $B \text{Cat}(\tilde{G}, \coprod GL_n(R))$.

**Question**: Is this an equivariant infinite loop space?
Step 2: Equivariant delooping

\( \text{Cat}(\tilde{G}, \coprod GL_n(R)) \) turns out to be an algebra over the genuine \( E_\infty \)-operad as defined by Guillou-May.
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$\mathcal{C}at(\tilde{G}, \coprod GL_n(R))$ turns out to be an algebra over the genuine $E_\infty$-operad as defined by Guillou-May.
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\( \text{Cat}(\tilde{G}, \coprod GL_n(R)) \) turns out to be an algebra over the genuine \( E_\infty \)-operad as defined by Guillou-May.
Equivariant Segal machine

Issue: There is another equivariant infinite loop space machine, the equivariant version of Segal’s machine, developed by Schimakawa.
Comparison theorem

Question: Do these equivariant infinite loop space machines produce equivalent outputs?

The proof of the nonequivariant comparison theorem of May-Thomason completely fails equivariantly!
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Theorem (May-M.-Osorno)
For an exact/Waldhausen $G$-category $C$, $\text{Cat}(\tilde{G}, C)$ is also exact/Waldhausen.

**Definition**

Define $K_G(C)$ to be $\Omega QBCat(\tilde{G}, C)$ if $C$ is an exact $G$-category and $\Omega|wS\bullet\text{Cat}(\tilde{G}, C)|$ if $C$ is a Waldhausen $G$-category.

**Theorem (M.)**

$+ = Q = S\bullet$.

**Conjecture**

$K_G(C)$ is an infinite loop $G$-space.
Future Directions

- $K$-theory of $G$-Waldhausen categories; do we have a splitting

$$A_G(M) \sim \Sigma_G^\infty M_+ \times Wh_G^{PL}(M),$$

where $Wh_G^{PL}(M)$ encodes information about equivariant pseudo-isotopies of a $G$-manifold $M$?

- Carlsson’s program for describing the $K$-theory of a field in terms of the representation theory of the absolute Galois group. (His map occurs as the fixed point map of the constructions I gave above.)
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Thank you!!!