HOPF FIBRATION

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1. Introduction

Covering spaces were introduced as useful tools in computing the fundamental groups of various topological spaces. We give here a quick review of the definition of covering spaces.

Definition 1.1. Let $p : E \to B$ be a continuous surjection. An open set $U \subseteq B$ is said to be ***evenly covered*** if its inverse image $p^{-1}(U)$ can be written as the disjoint union of open sets $V_\alpha \in E$, such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U.

Definition 1.2. If for every $b \in B$, there is an evenly covered open subset of B containing b, then p is a ***covering map***, and E is a ***covering space*** for B.

Notice that for every point $b \in B$, the subspace topology on $p^{-1}(b)$ is the discrete topology. Covering spaces can be generalised to the case in which the preimage $p^{-1}(b)$ is not discrete. To do this, we make the following definition:

Definition 1.3. A ***locally trivial fibre bundle*** is a quadruple (E, B, F, p) where E, B and F are topological spaces, and $p : E \to B$ is a map possessing the following properties:

(1) Any point $b \in B$ admits a neighbourhood U with the preimage $p^{-1}(U)$ being homeomorphic to $U \times F$.
(2) The homeomorphism $\psi : U \times F \to p^{-1}(U)$ is consistent with the map p, i.e. the following diagram commutes:

$$\begin{tikzcd} U \times F \arrow[rr, "\psi"] \arrow[dr, "\text{proj.}"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

Definition 1.4. In the locally trivial fibre bundle defined above, we say that E is the ***fibre space***, B is the ***base space***, F is the ***fibre***, and p is the ***projection*** or ***fibre map***.

Note that the pre-image of each point $b \in B$, $p^{-1}(b)$, is homeomorphic to F, since ψ restricted to $\{b\} \times F$ is a homeomorphism between $\{b\} \times F$ and $p^{-1}(b)$, and $\{b\} \times F$ is certainly homeomorphic to F.

Locally trivial fibre bundles as described in the above definitions is typically denoted as $F \to E \xrightarrow{p} B$. A fibre bundle shows that locally, E looks like the space $B \times F$, although this need not necessarily be true globally. Below, we give a few examples of locally trivial fibre bundles, or more simply, fibre bundles:

Example 1.5. For any space B, let $E = B \times F$. Then defining p as the projection map and ψ as the identity map, we clearly obtain a fibre bundle. In this case, E is both locally and globally homeomorphic to $B \times F$. Such fibre bundles are known as ***trivial***.

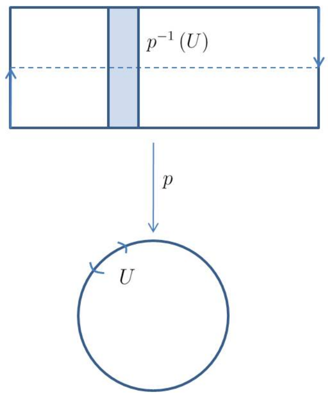

FIGURE 1.1. The Möbius band as the fibre space of S^1.

Example 1.6. Given a space B and a covering space E, every point $b \in B$ has an evenly covered open subset $U \subseteq B$ containing b. Now, $p^{-1}(U)$ can be expressed as the disjoint union of open subsets $V_\alpha \subseteq E$, with $\alpha \in A$ for some indexing set A. Then (E, B, A, p) form a fibre bundle, with the homeomorphism $\psi : U \times A \to E$ mapping $U \times \{\alpha\}$ to V_α. Hence every covering is a fibre bundle.

Example 1.7. The Möbius band is the fibre space of a bundle with base space S^1 and fibre I. Viewing the Möbius band as a rectangle with opposite sides of the rectangle glued together in the opposite orientation, the fibre map p maps the rectangle to the dashed line in Figure 1.1, which is homeomorphic to S^1. The open neighbourhood U of S^1 shown in Figure 1.1 has a strip in the rectangle as its preimage. This strip is clearly homeomorphic to $U \times I$.

2. The Hopf Fibration

One of the earliest example of a non-trivial fibre bundle was proposed by Heinz Hopf in 1931. The Hopf fibration defines a fibre map $p : S^3 \to S^2$ with fibres S^1, represented as $S^1 \hookrightarrow S^3 \xrightarrow{p} S^2$. This means that for every point $b \in S^2$, $p^{-1}(b) \in S^3$ is homeomorphic to S^1. When we define p later on, we shall show that $p^{-1}(b)$ turns out to be a great circle of S^3.

We first note that $S^3 = \{x, y, z, t \in \mathbb{R} | x^2 + y^2 + z^2 + t^2 = 1\}$ can be represented in $\mathbb{C}^2$ as $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | z_1\overline{z_1} + z_2\overline{z_2} = 1\}$. Next, we introduce the complex projective line, which will be crucial in developing the Hopf fibration.

Definition 2.1. Define an equivalence relation $\sim$ on the set $\mathbb{C}^2$, such that $(z_1, z_2) \sim (w_1, w_2)$ if $(z_1, z_2) = c \cdot (w_1, w_2)$ for some $c \in \mathbb{C} \setminus \{0\}$. Then the ***complex projective line***, denoted by $\mathbb{CP}^1$, is the set of equivalence classes in $\mathbb{C}^2 \setminus \{\mathbf{0}\}$, with the classes being denoted $[z_1 : z_2]$. Specifically, $\mathbb{CP}^1 = \{[z : 1] \, | z \in \mathbb{C}\} \cup [1 : 0]$.

Observe that $\mathbb{CP}^1$ is really the set of equivalence class in $S^3 \subseteq \mathbb{C}^2$, with antipodal points identified. The definition given is completely analogous to the definition of P^2, the real projective plane, as the antipodal points of S^2 identified.

Notice that $\mathbb{CP}^1$ has a copy of $\mathbb{C}$ plus an addition point. This suggests the following proposition:

Proposition 2.2. $\mathbb{CP}^1$ *is homeomorphic to* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Proof. $\mathbb{CP}^1$ has the quotient topology with $\mathbb{CP}^1 = S^3 / \sim$. We know from complex analysis that $\hat{\mathbb{C}}$ is homeomorphic to S^2, also known as the Riemann sphere, via stereographic projection. An

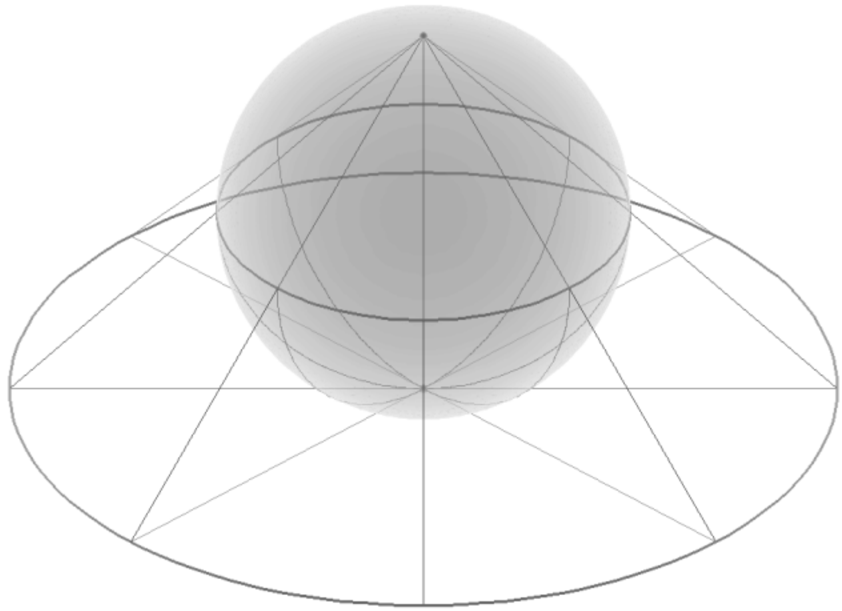

FIGURE 2.1. Stereographic projection, showing that $\hat{\mathbb{C}}$ is homeomorphic to S^2.

illustration of this is shown in Figure 2.1. Now consider the function $f : S^3 \to S^2$, which maps $(z_1, z_2) \in S^3$ to the point in S^2 corresponding to $z_1/z_2 \in \hat{\mathbb{C}}$. Once again, relying on results in complex analysis, f is a continuous function, since it just takes one complex number and divides it by another. Given that f is a continuous map from a compact space (S^3) to a Hausdorff space (S^2), f must be a quotient map. But for (z_1, z_2), $(w_1, w_2) \in S^3$, $f(z_1, z_2) = f(w_1, w_2)$ if and only if $z_1/z_2 = w_1/w_2$. But this implies that $(z_1, z_2) = c(w_1, w_2)$, and therefore $(z_1, z_2) \sim (w_1, w_2)$. Hence f is the quotient map which induces the equivalence relation $\sim$ on S^3. We must therefore conclude that $S^3/\sim$ is homeomorphic to S^2, i.e. $\mathbb{CP}^1$ is homeomorphic to $\hat{\mathbb{C}}$. $\square$

We are now ready to define the fibre map p in the proposed fibre bundle $S^1 \hookrightarrow S^3 \xrightarrow{p} S^2$. This is defined as the map $p : S^3 \to \mathbb{CP}^1$, $p : (z_1, z_2) \mapsto [z_1 : z_2]$, since $\mathbb{CP}^1$ is homeomorphic to $\mathbb{C} \cup \{\infty\}$ and therefore to S^2. Note that two points (z_1, z_2), $(z_1', z_2') \in S^3$ have the same image under p if and only if there exists λ such that $z_1' = \lambda z_1$, $z_2' = \lambda z_2$. But this means that

$$1 = z_1'\overline{z_1'} + z_2'\overline{z_2'} = \lambda\overline{\lambda}\left(z_1\overline{z_1} + z_2\overline{z_2}\right) = \lambda\overline{\lambda}$$

i.e. $|\lambda| = 1$. Thus, if $p(z_1, z_2) = [z_1 : z_2]$, then $p^{-1}[z_1 : z_2]$ consists of all those points which are obtained from (z_1, z_2) by multiplying each coordinate by $e^{i\theta}$, $-\pi < \theta \leq \pi$. This shows that the pre-image of each point on S^2 is a great circle on S^3. Therefore, $\forall b \in S^2$, $p^{-1}(b)$ is homeomorphic to S^1.

Now consider a covering of S^2, consisting of $S^2 - [0:1]$ and $S^2 - [1:0]$. Referring to Figure 2.1, and noting that $[0:1]$ corresponds to the point $\mathbf{0} \in \hat{\mathbb{C}}$, i.e. $\mathbf{0} \in S^2$, and $[1:0]$ corresponds to the point $\infty \in \hat{\mathbb{C}}$, we see immediately that $S^2 - [0:1]$ is really $S^2 - \{\mathbf{0}\}$, and $S^2 - [1:0]$ is equivalent to $\mathbb{R}^2$. These two sets $S^2 - \{\mathbf{0}\}$ and $\mathbb{R}^2$ therefore form an open covering of S^2.

We will now give a homeomorphism ψ_1 between $(S^2 - \{\mathbf{0}\}) \times S^1$ and $p^{-1}(S^2 - \{\mathbf{0}\})$, as well as a homeomorphism ψ_2 between $\mathbb{R}^2 \times S^1$ and $p^{-1}(\mathbb{R}^2)$. Defining $\mu \in S^2$ to correspond to the extended complex number z_1/z_2, we define

$$\psi_1(\mu,\theta) = \begin{cases} \left(\frac{e^{i\theta}}{\sqrt{1+\frac{1}{|\mu|^2}}}, \frac{e^{i\theta}}{\mu\sqrt{1+\frac{1}{|\mu|^2}}}\right), & \mu \neq \infty \\ \left(e^{i\theta}, 0\right), & \mu = \infty \end{cases}$$

and

$$\psi_2(\mu,\theta) = \left(\frac{\mu e^{i\theta}}{\sqrt{1+|\mu|^2}}, \frac{e^{i\theta}}{\sqrt{1+|\mu|^2}}\right)$$

It is a matter of complex analysis to determine that these two functions are indeed continuous, bijective and with continuous inverse. Hence we have shown that for every point $b \in S^2$, b admits a neighbourhood U, such that $p^{-1}(U)$, which can be made to be either a subset of $p^{-1}(S^2 - \{\mathbf{0}\})$ or $p^{-1}(\mathbb{R}^2)$, is homeomorphic to $U \times S^1$ through either of the homeomorphisms restricted to U. It is also readily apparent that $p \circ \psi_1(\mu,\theta) = \mu$ and $p \circ \psi_2(\mu,\theta) = \mu$ over their respective allowed μ. Therefore, we have successfully constructed the Hopf fibration, $S^1 \hookrightarrow S^3 \xrightarrow{p} S^2$.

3. A Very Brief Introduction to Quantum Mechanics

The Hopf fibration turns out to be fundamental to the consistency of quantum mechanics as a theory. To understand this deep connection between topology and physics, we need to give a brief overview of quantum mechanics.

Mechanics as a field in physics is really interested in the following question: given a set of particles numbered $i = 1, \cdots, n$, with known position and velocity vectors $\mathbf{x}_i$ and $\dot{\mathbf{x}}_i$ at a given time t, how can the position and velocity vectors of these same set of particles be predicted at some other time t'? This fundamental question is answered classically by describing how particles interact with each other, i.e. the forces between them, and how forces change the position and velocity vectors of each particle over time.

For various reasons, classical mechanics was found to be unsuitable as a description of nature in the microscopic limit. Instead, physics in the small scale turned out to be much less intuitive than classical mechanics. Physicists were essentially forced to accept some odd things about the true nature of reality in their attempts at formulating what has come to be known as quantum mechanics.

Quantum mechanics describes physical systems by a mathematical object known as a ***state***. These objects do not actually contain concrete information, such as position and velocity vectors, but rather tell us the various probabilities of observing all the physically possible results, if an observation were to be made. Crucially, if no observation is made, then the system cannot be said to be in any one state at all. This "uncertainty" is not simply a lack of information and understanding of the system, but really is a fundamental property of nature. Once an actual observation is made however, the state "collapses" into an actual physical state with known properties. The physical state observed, and the probability of measuring such a physical state in the first place depends entirely on the state.

3. The Spin-1/2 System

One of the cleanest examples of how quantum mechanics works is the spin-1/2 system. Spin is a vector that is a fundamental property of all particles in nature, much like mass, that affects how a particle behaves in a magnetic field. In all the particles that form everyday varieties of matter, the magnitude of this spin turns out to be 1/2.

When a spin-1/2 particle such as an electron is placed in a magnetic field, which is conventionally directed along the z-axis, we can perform a measurement to determine the component of its spin along the z-axis. When we do this, we find that there are only two possible outcomes. We can call these two configurations ***spin-up*** and ***spin-down***, denoted χ_+ and χ_-.

The state of this electron can be thought of as a "mixture" of χ_+ and χ_-. These two configurations are mutually exclusive outcomes of the measurement, with some probability p_+ and p_- of occurring respectively. This description immediately suggests the structure of a 2-dimensional vector space, with χ_+ and χ_- forming an orthonormal basis, and the state of the electron being a vector ψ, which is a linear combination of these basis elements.

For a quantum mechanical description of this system to make sense, we want to say that if two electrons have the same value for p_+ and p_-, they are in the same *quantum state*, even though they may in fact be in different *states*, i.e. with different mixtures of χ_+ and χ_-. This means that there must be an exact correspondence between the mathematical objects in theory known as quantum states and physical reality. Mathematically, we want to define quantum states of an electron as equivalence classes of states which yield the same physical results. This requires a careful definition of what is a state vector, and which state vectors are equivalent.

To do so formally, we envision the state vectors as vectors in the Hilbert space $\mathbb{C}^2$.

Definition 3.1. $\mathbb{C}^2$ is a ***Hilbert space***, or an ***inner product space***, over the field $\mathbb{C}$, equipped with an operator $\langle\,|\,\rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ known as an ***inner product***, which satisfies the following conditions:

(1) $\forall \psi, \varphi \in \mathbb{C}^2$, $\langle\psi|\varphi\rangle = \overline{\langle\varphi|\psi\rangle}$ (symmetry).
(2) $\forall \psi, \varphi, \zeta \in \mathbb{C}^2$ and $a, b \in \mathbb{C}$, $\langle a\psi + b\varphi|\zeta\rangle = \overline{a}\,\langle\psi|\zeta\rangle + \overline{b}\,\langle\varphi|\zeta\rangle$, and $\langle\psi|a\varphi + b\zeta\rangle = a\,\langle\psi|\varphi\rangle + b\,\langle\psi|\zeta\rangle$ (bilinearity)
(3) $\forall \psi \in \mathbb{C}^2$, $\langle\psi|\psi\rangle \in \mathbb{R} \subseteq \mathbb{C}$, and $\langle\psi|\psi\rangle \geq 0$, and $\langle\psi|\psi\rangle = 0$ if and only if $\psi = \mathbf{0}$. (positive definiteness)

For two state vectors $\psi = (\psi_1, \psi_2)$ and $\varphi = (\varphi_1, \varphi_2)$, the inner product is given explicitly as

$$\langle\psi|\varphi\rangle = \overline{\psi_1}\varphi_1 + \overline{\psi_2}\varphi_2$$

In $\mathbb{R}^n$, we can define the length of a vector using the regular scalar product. We can do the same in $\mathbb{C}^2$, with the concept of a norm.

Definition 3.2. The ***norm*** of the inner product space $\mathbb{C}^2$ is the operator $\|\|: \mathbb{C}^2 \to \mathbb{R}$ given by

$$\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$$

$\forall \psi \in \mathbb{C}^2$.

We are now ready to define the state, and the equivalence of states, and show that this definition is consistent with physical reality.

Definition 3.3. The ***state*** of an electron is defined as an element $\psi \in S(\mathbb{C}^2)$, where $S(\mathbb{C}^2) = \{\psi \in \mathbb{C}^2 | \, \|\psi\| = 1\}$. This means that $\psi = (\psi_1, \psi_2)$, with $|\psi_1|^2 + |\psi_2|^2 = 1$.

Definition 3.4. Two states $\psi, \varphi \in S(\mathbb{C}^2)$ are considered ***equivalent*** when $\exists \lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $\varphi = \lambda\psi$. The ***quantum state*** of an electron is the equivalence class

$$[\psi] = \{\lambda\varphi | \lambda \in \mathbb{C}, |\lambda| = 1\}$$

Now, for these definitions to be physically consistent, any two electrons in the same quantum state should give similar outcomes when a certain measurement is made. Since any measurement of an observable can lead to any one of many outcomes, what we really want is for the ***expectation value*** of these measurements to be the same. Quantum theory dictates that the spin-$1/2$ system with state ψ entirely determines this expectation value of an observable in the following way:

Proposition 3.5. *Every observable has an associated linear operator $\hat{S}$. The expectation value of the observable, $\overline{S}$, of a system in a particular state ψ is then given by*

$$\overline{S} = \left\langle \psi \middle| \hat{S}\psi \right\rangle$$

For the case of the spin-$1/2$ system, the relevant observables are the components of the spin vector, s_x, s_y and s_z.

Definition 3.6. The related linear operator of the measurement of the components of the spin vectors in the x-, y- and z- directions are given by

$$\hat{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively. These matrices are known collectively as the ***Pauli matrices***.

Example 3.7. We would do well with an example at this point. We will consider the measurement of the spin vector along the z-axis of a electron with state

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The expectation value $\overline{s_z}$ is then

$$\begin{aligned} \overline{s_z} &= \left\langle \psi \middle| \hat{S}_z\psi \right\rangle \\ &= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\ &= 1 \end{aligned}$$

This means that we will always find the electron to be in the spin-up state! Clearly, the state vector ψ represents an electron which is purely in this state. Earlier, we motivated the use of a 2-dimensional vector space to describe the state vector by saying that each state can be viewed as a mixture of χ_+ and χ_-. Now, we have the natural identification

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_+, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_-$$

We can therefore view ψ as a linear combination of χ_+ and χ_-, i.e. $\psi = \psi_1\chi_+ + \psi_2\chi_-$.

Example 3.8. Again, we have the state

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

If we now make a measurement of the y-component of the spin, we get the expectation value

$$\begin{aligned}
\overline{s_y} &= \left\langle \psi \middle| \hat{S}_y \psi \right\rangle \\
&= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ i \end{pmatrix} \right\rangle \\
&= 0
\end{aligned}$$

An expectation value of zero suggests that we have the probability of finding the spin in the y-component to be spin-up or spin-down is equal. This shows that if we are in a state of definite spin in the z-direction, the spin in the y-direction is completely indefinite.

Thus, we hope to show two electrons in the same quantum state $[\psi]$ has the same expectation value for the spin vector $\overline{\mathbf{s}} = \overline{s_x}\hat{x} + \overline{s_y}\hat{y} + \overline{s_z}\hat{z}$. The total spin for the electron is $1/2$, but in defining the Pauli matrices this has been normalised to 1. We now show that the Hopf fibration indeed guarantees that this is the case.

Theorem 3.9. *There exists a one-to-one correspondence between quantum states and expectation values of the spin vector.*

Proof. States are defined on the space $S(\mathbb{C}^2)$. This clearly is homeomorphic to the space S^3. The spin vectors also clearly correspond to points on the sphere S^2. Now, consider the map $f : S^3 \to S^2$ which maps $f : \psi \mapsto \overline{\mathbf{s}}(\psi)$. Then

$$\begin{aligned}
f\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \left\langle \psi \middle| \hat{S}_x \psi \right\rangle \hat{x} + \left\langle \psi \middle| \hat{S}_y \psi \right\rangle \hat{y} + \left\langle \psi \middle| \hat{S}_z \psi \right\rangle \hat{z} \\
&= \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} \right\rangle \hat{x} + \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{pmatrix} -i\psi_2 \\ i\psi_1 \end{pmatrix} \right\rangle \hat{y} + \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} \right\rangle \hat{z} \\
&= \left(\overline{\psi_1}\psi_2 + \overline{\psi_2}\psi_1\right)\hat{x} + i\left(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2\right)\hat{y} + \left(\overline{\psi_1}\psi_1 - \overline{\psi_2}\psi_2\right)\hat{z}
\end{aligned}$$

Now, points (ξ, η, ζ) on S^2 corresponds to a point on $\hat{\mathbb{C}}$ via the stereographic projection described in Proposition 2.2.[1] Denote this point by $z = x + iy$. Then

[1]The map given below isn't precisely the stereographic projection in Figure 2: (ξ, η, ζ) gets sent to $-y$ instead of y if we use the map in Figure 2.1. But this does not change the fact that there is a one-to-one correspondence between S^2 and $\hat{\mathbb{C}}$.

$$\begin{aligned} x &= \frac{\xi}{1-\zeta} \\ &= \frac{\overline{\psi_1}\psi_2 + \overline{\psi_2}\psi_1}{1-\overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2} \\ &= \frac{\overline{\psi_1}\psi_2 + \overline{\psi_2}\psi_1}{2\overline{\psi_2}\psi_2} \\ &= \frac{1}{2}\left(\frac{\overline{\psi_1}}{\overline{\psi_2}} + \frac{\psi_1}{\psi_2}\right) \\ &= \mathrm{Re}\left(\frac{\psi_1}{\psi_2}\right) \end{aligned}$$

$$\begin{aligned} y &= -\frac{\eta}{1-\zeta} \\ &= -\frac{i\left(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2\right)}{1-\overline{\psi_1}\psi_1 + \overline{\psi_2}\psi_2} \\ &= -\frac{i\left(\overline{\psi_2}\psi_1 - \overline{\psi_1}\psi_2\right)}{2\overline{\psi_2}\psi_2} \\ &= -\frac{i}{2}\left(\frac{\psi_1}{\psi_2} - \frac{\overline{\psi_1}}{\overline{\psi_2}}\right) \\ &= \mathrm{Im}\left(\frac{\psi_1}{\psi_2}\right) \end{aligned}$$

Thus $f : (\psi_1, \psi_2) \mapsto \psi_1/\psi_2$ when S^2 is viewed as the extended complex plane is precisely the quotient map $f : S^3 \to S^2$ defined in Proposition 2.2. But note that f is completely equivalent to the map $p : S^3 \to \mathbb{CP}^1$, $p : (z_1, z_2) \to [z_1 : z_2]$, since $\mathbb{CP}^1$ is the quotient space of S^3 with $(z_1, z_2) \sim (w_1, w_2)$ if and only if $(z_1, z_2) = c\,(w_1, w_2)$, which is true if and only if $z_1/z_2 = w_1/w_2$. Hence there is a one to one correspondence between ψ_1/ψ_2 and $[\psi_1 : \psi_2]$ in $\mathbb{CP}^1$. Hence the map $f : \psi \mapsto \bar{\mathbf{s}}\,(\psi)$ is the fibre map p of the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{p} S^2$. As we noted earlier, two points $\psi, \varphi \in S^3$ have the same image under f if and only if $\psi = \lambda\varphi$, where $|\lambda| = 1$. But this is precisely the same as saying that ψ, φ have the same image when they are in the same equivalence class $[\psi]$, i.e. they represent the same quantum state. This establishes a one-to-one correspondence between quantum states on S^3 and expectation value of the spin vector in S^2, as required. □

4. Conclusion

The result obtained in this paper is significant, as it establishes a link between the abstraction of quantum theory and physical reality, which quantum theory was ultimately designed to describe. We have, however, only shown this link for the spin-½ system. This system, also known as a ***qubit***, is the simplest system within the quantum theoretical framework. Higher dimensional analogues of this system exist, and a particularly important example is the two-qubit, a system which exhibits ***entanglement***.

The phenomenon of entanglement describes a correlation between the states of two particles. Simply put, in a pair of entangled particles, measuring the state of one particle (which causes the state vector to collapse) instantly determines the state of the other. This would happen *even if these two particles were light-years apart*, with no possibility of interaction. This is a profoundly counter-intuitive behaviour that has actually been experimentally observed.

It turns out that a higher dimensional generalisation of the Hopf fibration, which produces the fibre bundle $S^3 \hookrightarrow S^7 \xrightarrow{p} S^4$, provides the link between quantum mechanical theory and the physical reality of the two-qubit. Evidently, the Hopf fibration is crucial to our understanding of these systems.

It is somewhat surprising that such an abstract concept in algebraic topology can have such an unexpected and profound impact on physical reality. But this is not an isolated case: physical theory is intricately tied to results from all fields of mathematics. This ability of mathematics to even begin to describe the real world has often been viewed by physicists as wondrous and inexplicable. If so, then the Hopf fibration is a marvellous glimpse at a miracle of nature.

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