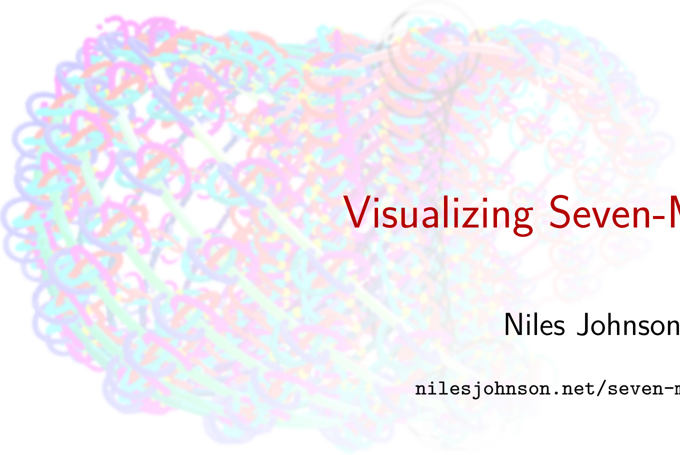

Visualizing Seven-Manifolds

Niles Johnson

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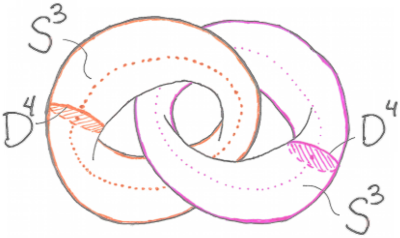

Visualizing Seven-Manifolds

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In 1956 John Milnor startled the mathematical community by constructing smooth 7-dimensional manifolds that are homeomorphic but not diffeomorphic to the standard seven-sphere. His discovery opened a new branch of research in topology and won him the Fields Medal in 1962.

Milnor, John. *On Manifolds Homeomorphic to the 7-Sphere*.
Annals of Mathematics Vol. 64 No. 2. September 1956.

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These manifolds are fibered over the four-sphere, with three-sphere fibers

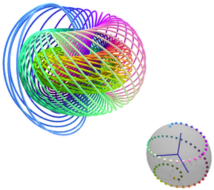
$$S^3 \to M \to S^4.$$

They are the unit-sphere bundles of $\mathbb{R}^4$ bundles over S^4. As such they are classified by structure maps

$$S^3 \to SO(4).$$

$$\pi_3\big(SO(4)\big) \cong \mathbb{Z} \times \mathbb{Z}$$

$$\xi_{h,j} \leftrightarrow (h, j)$$



S^3, organized by Hopf fibers over S^2

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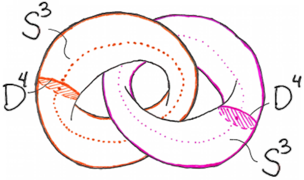
Given $(h, j) \in \mathbb{Z} \times \mathbb{Z}$, the total space M is produced by gluing together two copies of $S^3 \times D^4$ along their boundary.

$$M = M(h, j) = \left(S^3 \times D^4\right) \cup_{\xi_{h,j}} \left(S^3 \times D^4\right)$$

The attaching map $\xi_{h,j}$ uses quaternion multiplication, regarding $\partial D^4 = S^3 \subset \mathbb{H}$:

$$\begin{aligned} S^3 \times \partial D^4 &\xrightarrow{\xi_{h,j}} S^3 \times \partial D^4 \\ (u, v) &\longmapsto (u,\ u^h\, v\, u^j\,) \end{aligned}$$



Glue two copies of $S^3 \times D^4$ along their boundary.

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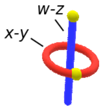
At each point u of S^3

$$\xi_{h,j}(u,-) : \partial D^4 \to \partial D^4$$

is a diffeomorphism of ∂D^4.

We depict this boundary as the stereographic projection of two reference circles in the unit quaternions,

$$\partial D^4 = \left\{(w,x,y,z) \in \mathbb{H} \;\middle|\; w^2+x^2+y^2+z^2 = 1\right\}$$



one in the w-z plane and the other in the x-y plane.

These are the cores of two solid tori whose union is ∂D^4.

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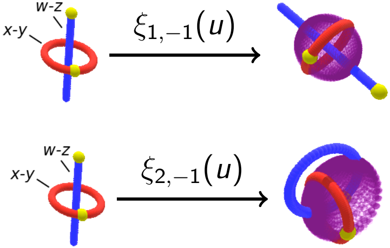
For each point $u \in S^3$, we depict the diffeomorphism $\xi_{h,j}(u, -)$ by drawing the images of the reference circles under the map. We also draw the image of the two-sphere $w = 0$ (purple).



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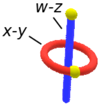
When $h + j = 0$,

$$H^*(M) = \begin{cases} \mathbb{Z} & * = 0, 3, 4, 7 \\ 0 & \text{else} \end{cases}$$

However, $M = M(h, -h)$ is homeomorphic to $S^3 \times S^4$ if and only if $h = 0$.

The map $\xi_{h,-h}(u)$ is quaternion conjugation by u^h and therefore fixes -1 and $+1$. These lie on the *w-z* reference circle, and under stereographic projection they map to the center and boundary of D^3, respectively.



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When $h + j = 1$, M is homeomorphic to S^7 but not necessarily of the same diffeomorphism type. A diffeomorphism invariant is given by:

$$\lambda := (h - j)^2 - 1 \pmod 7$$
$$k := h - j$$

$h + j = 1$	λ	M_k	
$\xi_{1,0}$	0	S^7_1	standard
$\xi_{2,-1}$	1	S^7_3	*exotic*
$\xi_{3,-2}$	3	S^7_5	*exotic*

$h + j = 1$	λ	M_k	
$\xi_{4,-3}$	6	S^7_7	*exotic*
$\xi_{6,-5}$	1	S^7_{11}	*exotic*
$\xi_{8,-7}$	0	S^7_{15}	*exotic?*

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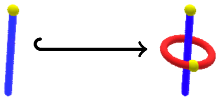
To compare diffeomorphism types of seven-spheres, we focus on the *w*-*z* reference circle.



Restricting to this circle, $\xi^*_{h,j}$ is a map from S^3 to the space of embedded circles in $S^3 = \partial D^4$.

$$\xi^*_{h,j} \colon S^3 \xrightarrow{\xi_{h,j}} \mathrm{Diff}(S^3) \xrightarrow{\text{restr. to } (w,0,0,z)} \mathrm{Emb}(S^1 \hookrightarrow S^3)$$

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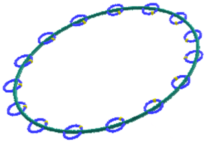
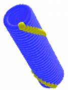
Stacking up the embedded circles along a Hopf fiber results in an embedded tube. Twisting of this embedding indicates the difficulty of finding a diffeomorphism between two separate seven-spheres.

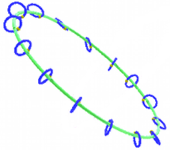
$$S^1 \xrightarrow{\text{Hopf fiber}} S^3 \xrightarrow{\xi^*_{h,j}} \mathrm{Emb}(S^1 \hookrightarrow S^3)$$

$\xi^*_{1,0}$:





$\xi^*_{2,-1}$:



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Now let's take a look at a few of the seven-manifolds constructed by Milnor! We'll consider bundles $M(h, -h)$ followed by standard and exotic seven-spheres.

We show $\xi_{h,j}(u, -)$ for various u along a pair of Hopf fibers in S^3. One point u is circled, and we draw a closeup version of $\xi_{h,j}$ at that point.