Visualizing Seven-Manifolds

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In 1956 John Milnor startled the mathematical community by constructing smooth 7-dimensional manifolds that are homeomorphic but not diffeomorphic to the standard seven-sphere. His discovery opened a new branch of research in topology and won him the Fields Medal in 1962.

These manifolds are fibered over the four-sphere, with three-sphere fibers

\[ S^3 \to M \to S^4. \]

They are the unit-sphere bundles of \( \mathbb{R}^4 \) bundles over \( S^4 \). As such they are classified by structure maps

\[ S^3 \to SO(4). \]

\[ \pi_3(SO(4)) \cong \mathbb{Z} \times \mathbb{Z} \]

\( \xi_{h,j} \leftrightarrow (h,j) \)

\( S^3 \), organized by Hopf fibers over \( S^2 \)
Given $(h, j) \in \mathbb{Z} \times \mathbb{Z}$, the total space $M$ is produced by gluing together two copies of $S^3 \times D^4$ along their boundary.

$$M = M(h, j) = (S^3 \times D^4) \cup_{\xi_{h,j}} (S^3 \times D^4)$$

The attaching map $\xi_{h,j}$ uses quaternion multiplication, regarding $\partial D^4 = S^3 \subset \mathbb{H}$:

$$S^3 \times \partial D^4 \xrightarrow{\xi_{h,j}} S^3 \times \partial D^4 \quad (u, v) \mapsto (u, u^h v u^i)$$

Glue two copies of $S^3 \times D^4$ along their boundary.
At each point $u$ of $S^3$\[\xi_{h,j}(u,-) : \partial D^4 \rightarrow \partial D^4\]
is a diffeomorphism of $\partial D^4$.

We depict this boundary as the stereographic projection of two reference circles in the unit quaternions,\[
\partial D^4 = \{(w, x, y, z) \in \mathbb{H} \mid w^2 + x^2 + y^2 + z^2 = 1\}
\]
one in the $w$-$z$ plane and the other in the $x$-$y$ plane.

These are the cores of two solid tori whose union is $\partial D^4$. 
For each point \( u \in S^3 \), we depict the diffeomorphism \( \xi_{h,j}(u, -) \) by drawing the images of the reference circles under the map. We also draw the image of the two-sphere \( w = 0 \) (purple).
When \( h + j = 0 \),

\[
H^*(M) = \begin{cases} 
\mathbb{Z} & * = 0, 3, 4, 7 \\
0 & \text{else}
\end{cases}
\]

However, \( M = M(h, -h) \) is homeomorphic to \( S^3 \times S^4 \) if and only if \( h = 0 \).

The map \( \xi_{h,-h}(u) \) is quaternion conjugation by \( u^h \) and therefore fixes \(-1\) and \(+1\). These lie on the \( w-z \) reference circle, and under stereographic projection they map to the center and boundary of \( D^3 \), respectively.
When \( h + j = 1 \), \( M \) is homeomorphic to \( S^7 \) but not necessarily of the same diffeomorphism type. A diffeomorphism invariant is given by:

\[
\lambda := (h - j)^2 - 1 \pmod{7}
\]

\[
k := h - j
\]

<table>
<thead>
<tr>
<th>( h + j = 1 )</th>
<th>( \lambda )</th>
<th>( M_k )</th>
<th>( h + j = 1 )</th>
<th>( \lambda )</th>
<th>( M_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_{1,0} )</td>
<td>0</td>
<td>( S_1^7 ) standard</td>
<td>( \xi_{4,-3} )</td>
<td>6</td>
<td>( S_7^7 ) exotic</td>
</tr>
<tr>
<td>( \xi_{2,-1} )</td>
<td>1</td>
<td>( S_3^7 ) exotic</td>
<td>( \xi_{6,-5} )</td>
<td>1</td>
<td>( S_{11}^7 ) exotic</td>
</tr>
<tr>
<td>( \xi_{3,-2} )</td>
<td>3</td>
<td>( S_5^7 ) exotic</td>
<td>( \xi_{8,-7} )</td>
<td>0</td>
<td>( S_{15}^7 ) exotic?</td>
</tr>
</tbody>
</table>
To compare diffeomorphism types of seven-spheres, we focus on the $w$-$z$ reference circle.

Restricting to this circle, $\xi_{h,j}$ is a map from $S^3$ to the space of embedded circles in $S^3 = \partial D^4$.

$$\xi_{h,j}^* : S^3 \xrightarrow{\xi_{h,j}} \text{Diff}(S^3) \xrightarrow{\text{restr. to } (w,0,0,z)} \text{Emb}(S^1 \hookrightarrow S^3)$$
Stacking up the embedded circles along a Hopf fiber results in an embedded tube. Twisting of this embedding indicates the difficulty of finding a diffeomorphism between two separate seven-spheres.

\[
S^1 \xrightarrow{\text{Hopf fiber}} S^3 \xrightarrow{\xi_{h,j}} \text{Emb}(S^1 \hookrightarrow S^3)
\]

\[
\xi_{1,0}^*: \\
\xi_{2,-1}^*:
\]
Now let’s take a look at a few of the seven-manifolds constructed by Milnor! We’ll consider bundles $M(h, -h)$ followed by standard and exotic seven-spheres.

We show $\xi_{h,j}(u, -)$ for various $u$ along a pair of Hopf fibers in $S^3$. One point $u$ is circled, and we draw a closeup version of $\xi_{h,j}$ at that point.