The 2-dimensional stable homotopy hypothesis

OR

Introduction to twice-categorified abelian groups

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What we proved

Theorem: There is an equivalence of homotopy theories
(categories with weak equivalences)

Stable 2-types $\simeq$ Picard 2-categories

Stable 2-types: w/ stable weak equiv.s

stable under suspension
- group comp'n of π_0
- abelianization of π_0, π_1 (E_∞ str.)

$\left[\pi_0, \pi_1, \pi_2 \text{ \& } 2 \text{ Postnikov inv'ts}\right]$

Picard 2-categories: w/ categorical equiv.s

[Permutative Gray Monoids w/ obj. & mor. invertible]

What we learned

- The strictest kind of SM bicategory which encodes fully general homotopy of stable 2-types is monoid w.r.t. Gray product on 2Cat.

PGM:
- 2-category
- Gray monoid
- permutative (strictly associative & unital)

$a \xrightarrow{f} a'$, $b \xrightarrow{g} b'$

$$\begin{tikzcd} \cdot \arrow[r, "{(f,1)}"] \arrow[d, "{(1,g)}"'] \arrow[dr, phantom, "\Sigma \swarrow"] & \cdot \arrow[d, "{(1,g)}"] \\ \cdot \arrow[r, "{(f,1)}"'] & \cdot \end{tikzcd}$$

$(1,g)(f,1) \overset{\Sigma_{f,g}}{\cong} (f,1)(1,g)$

Postnikov tower

$$\begin{tikzcd} & X_2 \arrow[d] & \\ \Sigma^1 \pi_1 \arrow[r, "i_1"] & X_1 \arrow[r, "k_1"] \arrow[d] & \Sigma^3 \pi_2 \\ \Sigma^0 \pi_0 \cong & X_0 \arrow[r, "k_0"] & \Sigma^2 \pi_1 \end{tikzcd}$$

Context:
- Stable 0-types $\simeq$ Abelian groups
- Stable 1-types $\simeq$ Picard categories [grouplike SM groupoids a.k.a. abelian 2-groups]

$k_0 : \pi_0/2 \to \pi_1$ $\leftrightsquigarrow$ symm $e \cong x \cdot x^{-1} \cong x^{-1} \cdot x \cong e$

$P_1 S$ $\leftrightsquigarrow$ $\mathrm{Pic}(\mathrm{Ch}(\underline{Ab}))$
obj. $\cong \mathbb{Z}$
aut $\cong \mathbb{Z}/2$

- infinite loop space machine for PGM (K-thy paper)
- homotopy theory w/ strict v.s. weak maps (homotopy theories & adjunctions)
- $k_1 i_1$ given by $\Sigma_{f,f'}$ (Postnikov data paper)
- SM bicat structure from E_∞ operad action (SHH2)

Next Steps

Homotopy theory of Picard 2-categories
- Homological algebra of Picard 1-categories
- 2-categorical group-completion (Quillen's $S^{-1}S$)
- homotopy fiber/cofiber via comma constr. (& Quillen B)

Questions

- $P_2 S \overset{?}{\simeq} \mathrm{Pic}(\mathrm{Ch}(\underline{\mathrm{Pic}}))$
  (2-category of Picard 1-categories)
  - categorification of $\otimes$
  - correct π_0, π_1, π_2 & k_0, $k_1 i_1$
  - need SM bicat str.
- $d_3 : \ker(k_0) \xrightarrow{?} \mathrm{coker}(k_1 i_1)$
  we have 2 guesses