Visualizing the Hopf fibration

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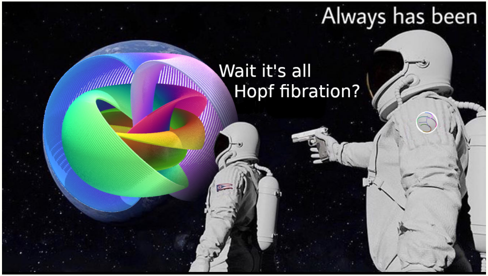
Introduction: The Hopf fibration

Fiber bundle structure for S^3 over S^2

Fundamental example of many phenomena in topology and geometry



Outline

Fiber bundles
Quaternion arithmetic
Visualizing

Spheres

Quick review of $\mathbb{R}^n$, S^n

Three ways to think about spheres:

- Unit sphere in Euclidean space
- Two hemispheres glued together
- Disk with boundary collapsed to point

Fiber bundles: recall cartesian products

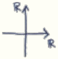
A **fiber bundle** is a “twisted” cartesian product.

For sets A and B, their **cartesian product** is

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

Examples

- $\mathbb{R} \times \mathbb{R}$



- $\mathbb{R} \times S^1$

- $S^1 \times S^1$

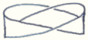
Projection

$$A \times B \to B$$

Fiber bundles: generalized projections

fiber		total space		base
F	$\longrightarrow$	E	$\longrightarrow$	B

Examples

- Möbius band 
- Hopf band 

The Hopf fibration is a fiber bundle of spheres!

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$

Note: A fiber bundle is a special kind of fibration; we won't discuss more general fibrations here.

Quaternion arithmetic

First recall complex numbers, $\mathbb{C}$:

$$x + y\,\mathbf{i} \text{ with } x, y \in \mathbb{R} \qquad \mathbf{i}^2 = -1$$

Quaternions, $\mathbb{H}$:

$$w + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \text{ with } w, x, y, z \in \mathbb{R}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{i}\mathbf{j} = \mathbf{k} \qquad \mathbf{j}\mathbf{k} = \mathbf{i} \qquad \mathbf{k}\mathbf{i} = \mathbf{j}$$

$$\mathbf{j}\mathbf{i} = -\mathbf{k} \qquad \mathbf{k}\mathbf{j} = -\mathbf{i} \qquad \mathbf{i}\mathbf{k} = -\mathbf{j}$$

S^3 = unit quaternions

$$\{q = w + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \mid w^2 + x^2 + y^2 + z^2 = 1\}$$

Quaternion arithmetic: the Hopf map

For a unit quaternion $q = w + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, you can check

$$q^{-1} = w - x\,\mathbf{i} - y\,\mathbf{j} - z\,\mathbf{k}$$

Define $\eta(q) = q\,\mathbf{k}\,q^{-1}$

$$= 0 + (2wy + 2xz)\,\mathbf{i} + (2yz - 2wx)\,\mathbf{j} + (w^2 + z^2 - x^2 - y^2)\,\mathbf{k}$$

You can check: $\eta(q)$ also has unit length, so it is in S^2 !

Fiber over a point $(a, b, c) \in S^2$ is

$$\{q \in S^3 \mid \eta(q) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}\}$$

Visualizing: stereographic projection

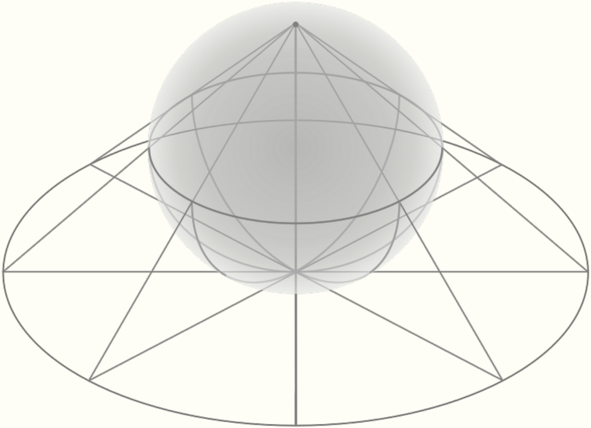


Image: Wikimedia CheChe (2017) [1]

Visualizing: stereographic projection

$$S^1 \longrightarrow \mathbb{R}^1 \cup \{\infty\} \cong D^1 \cup \{\infty\}$$

$$S^2 \longrightarrow \mathbb{R}^2 \cup \{\infty\} \cong D^2 \cup \{\infty\}$$

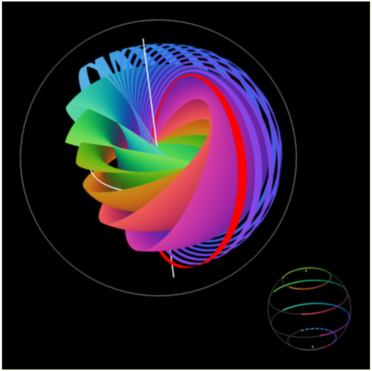
$$S^3 \longrightarrow \mathbb{R}^3 \cup \{\infty\} \cong D^3 \cup \{\infty\}$$

Image: Wikimedia Cmglee (2016) [2]

Visualizing: the Hopf fibration

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$

$$S^1 \longrightarrow D^3 \cup \{\infty\} \longrightarrow S^2$$

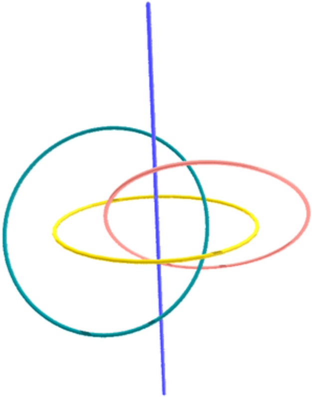


Let's watch together: `https://www.youtube.com/watch?v=AKotMPGFJYk`

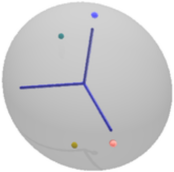
Image: Ken Shoemake (1997) [3]

Visualizing: the Hopf fibration

$S^3 \cong D^3 \cup \{\infty\}$
circles



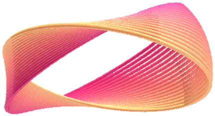
S^2
points



Visualizing: the Hopf fibration

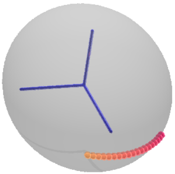
$S^3 \cong D^3 \cup \{\infty\}$

Hopf band



S^2

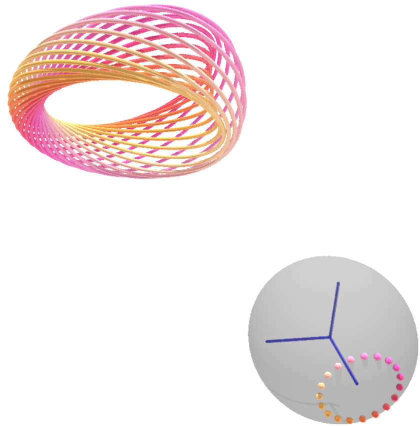
arc



Visualizing: the Hopf fibration

$S^3 \cong D^3 \cup \{\infty\}$
torus

S^2
circle



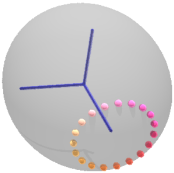
Visualizing: the Hopf fibration

$S^3 \cong D^3 \cup \{\infty\}$

solid torus

S^2

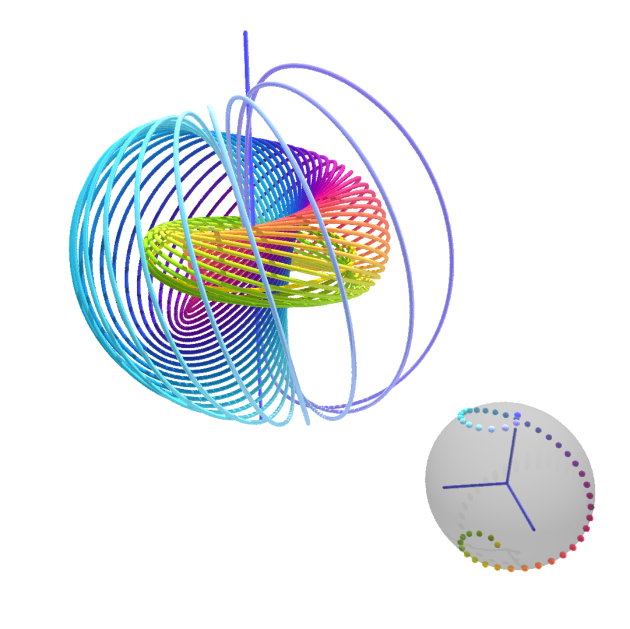
disk



Visualizing: the Hopf fibration

$S^3 \cong D^3 \cup \{\infty\}$
surfaces

S^2
curves



Visualizing: the Hopf fibration

Things to watch for:

- S^3 is a union of two solid torii, joined along their boundary
- A torus can be turned inside out in S^3 without intersecting itself
- The Hopf link is fibered: has a family of surfaces whose boundaries are the link, and are parametrized by a circle

Challenge questions:

- Explain why every pair of Hopf fibers is linked (with linking number 1).
- Explain why the Hopf map is not null-homotopic.

Let's watch now!

Thanks for this chance to talk about some cool mathematics!

I'm happy to answer any questions.

Credits and References

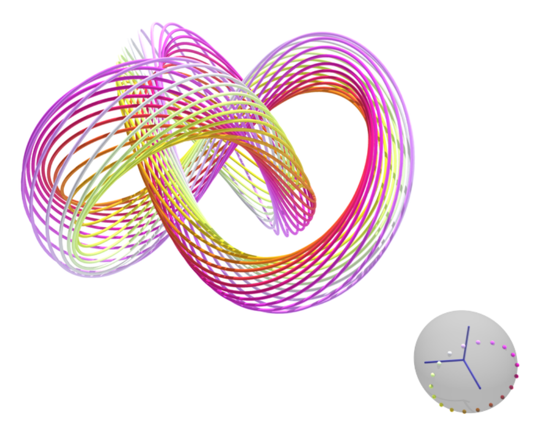
[1] Wikimedia User CheChe. Stereographic_projection_in_3D.svg (CC BY-SA 4.0).

[2] Wikimedia User Cmglee. Cmglee_Wikimania2016_Esino_Lario_Last_Supper_tinyplanet.jpg (CC BY-SA 4.0).

[3] Ken Shoemake (1997) https://hopf.math.purdue.edu/new-html/hopflogo.html

- Images computed and rendered with Sage: https://sagemath.org
- Further production notes: https://nilesjohnson.net/hopf-production.html
- Code for drawing Hopf fibers: https://github.com/nilesjohnson/hopf_fibration

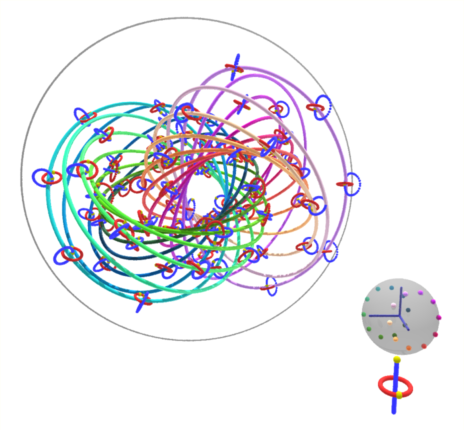
Conclusion: other things

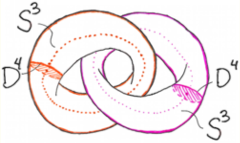


The modular fibration; visualization by Ihechukwu Chinyere

https://www.youtube.com/watch?v=eqeqbjec97w

Conclusion: other things





Exotic 7-spheres; based on work of Milnor

`https://www.youtube.com/watch?v=II-maE5HEj0`