Algebraic *K*-theory for 2-categories

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joint work with

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Abstract

Quillen recognized the higher algebraic K-groups of a ring R as homotopy groups of a certain topological space, $BGL(R)^+$. We review some of the basic definitions and computations via categorical algebra. We then describe how a 2-categorical extension of this theory leads to a new model for $K_3(R)$, together with more general applications. We will give a mild sampling of key technical details and close with some of the problems we're currently working on. The work we present is joint with Gurski-Osorno, Fontes-Gurski and Parab.

Outline

- Algebraic K-groups of a ring
  - Low-dimensional cases
  - Quillen's higher K-groups
  - Application for K_2
- 2-categorical analogue
  - Examples of symmetric monoidal 2-categories
  - Generalizations of Quillen's constructions
  - Application for K_3

Algebraic K-groups of a ring: $K_0(R)$

Ring R
$\mathrm{Proj}_{f.g.}(R)$ = finitely-generated projective R-modules
(symmetric monoidal category under $\oplus$.)

Definition

$$K_0(R) = Gr(\mathrm{Proj}_{f.g.}(R)/\cong).$$

(Gr = group-completion; known as Grothendieck construction)

Example for field F, $K_0(F) = \mathbb{Z}$.
Example Application $\det\colon K_0(R) \to \mathrm{Pic}(R)$
is a surjection.

Algebraic K-groups of a ring: $K_0(R)$

Generalize to Grothendieck group of other symmetric monoidal categories S:

Definition

$$K_0(S) = Gr(S/\cong).$$

Examples:

- $\mathrm{Proj}_{f.g.}(R)$
- vector bundles on topological spaces
- representations of finite groups

Algebraic K-groups of a ring: $K_1(R)$

Let $GL(R)$ be the infinite general linear group

$$GL(R) = \mathrm{colim}\Big(\cdots \hookrightarrow GL_n(R) \hookrightarrow GL_{n+1}(R) \hookrightarrow \cdots \Big).$$

(union of groups)

Definition

$$K_1(R) = GL(R)^{\mathrm{ab}} = GL(R) \Big/ [GL(R), GL(R)].$$

Equivalently,

$$K_1(R) = H_1(GL(R)).$$

Algebraic K-groups of a ring: $K_1(R)$

Theorem (Localization exact sequence)

Let S be a multiplicatively closed set of central elements in R. Then there is an exact sequence

$$K_1(R) \to K_1(S^{-1}R) \to K_0(\mathbf{Ft}_S(R)) \to K_0(R) \to K_0(S^{-1}R).$$

($\mathbf{Ft}_S(R)$ = S-torsion R-modules with finite-length projective resolutions)

Theorem (Fundamental Theorem)

There is an exact sequence

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t, t^{-1}]) \to K_0(R) \to 0.$$

Algebraic K-groups of a ring: $K_2(R)$

Lemma (Whitehead)

$$[GL(R), GL(R)] = E(R) = \mathrm{colim}_{\mathrm{n}} E_n(R)$$

where $E_n(R)$ is the elementary group generated by elementary $n \times n$ matrices.

(ones on diagonal; single off-diagonal entry)

Equivalent definitions

$$K_1(R) = GL(R)^{\mathrm{ab}} = GL(R)/E(R) = H^1(GL(R))$$

Algebraic K-groups of a ring: $K_2(R)$

The Steinberg group, $St(R)$ is generated by formal symbols $x_{ij}(r)$ for $r \in R$, subject to elementary relations.

(products and commutators of elementary matrices)

Note: $E(R)$ generally has more relations than $St(R)$.

Definition

$$K_2(R) = \ker\big(St(R) \to E(R) \big).$$

Theorem (Steinberg)

$K_2(R)$ is the center of $St(R)$. In particular, $K_2(R)$ is abelian.

Theorem (Bass)

$$K_2(R) \cong H_2(E(R)).$$

Properties of low-dimensional K-groups

Looking at the definitions, it may be unclear that the groups $K_i(R)$ are part of any reasonable sequence. Here are some clues.

- Localization exact sequence
- Fundamental Theorems for $R[t, t^{-1}]$
- $K_1(R)$ and $K_2(R)$ are modules over $K_0(R)$
- Product $K_1(R) \otimes K_1(R) \to K_2(R)$.

Quillen's higher K-groups

Definition

$$K_n(R) = \pi_n\left(BGL(R)^+\right), \quad n > 0$$

Explanation (Part 1) BG is the classifying space of a group G. (we will take $G = GL(R)$)

BG is the base of a principal G-bundle

$$G \to EG \to BG$$

- EG contractible and has free G-action
- therefore $BG \simeq K(G, 1)$, an Eilenberg-Mac Lane space

Example: $B\mathbb{Z} = S^1$

$$\mathbb{Z} \to \mathbb{R} \to S^1$$

Quillen's higher K-groups

Definition

$$K_n(R) = \pi_n\big(BGL(R)^+\big), \quad n > 0$$

Explanation (Part 2) The plus construction X^+ on a topological space X has two properties:

- $\pi_1(X^+) = \pi_1(X)^{ab} \cong H_1(X)$
- a map $X \to X^+$ inducing a homology isomorphism

Thus we certainly have

$$\pi_1(BGL(R)^+) \cong GL(R)^{ab} = K_1(R).$$

Quillen's higher K-groups

Definition

$$K_n(R) = \pi_n\big(BGL(R)^+\big), \quad n > 0$$

Explanation (Part 3) Why is this definition such a good one? (partial answer)

- Extends localization exact sequence and fundamental theorem.
- Explains graded ring structure on $K_n(R)$.
- Explains connection between algebraic K-groups and homotopy theory.

Quillen's higher K-groups: Second version

Let $\mathbb{Q} = \mathbb{Q}(\mathrm{Proj}_{f.g.}(R))$ be the following category.

- Objects are those of $\mathrm{Proj}_{f.g.}(R)$
- Morphisms $X \to Y$ are injection/surjection spans

$$\begin{array}{ccccc} & & S & & \\ & \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow & \\ X & & & & Y \end{array}$$

- Composition is via pullback

$$\begin{array}{ccccccccc} & & & & P & & & & \\ & & & \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow & & & \\ & & S & & & & T & & \\ & \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow & & \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow & \\ X & & & & Y & & & & Z \end{array}$$

Quillen's higher K-groups: Second version

Definition (Alternate)

$$K_n(R) = \pi_{n+1}(B\mathcal{Q}) \cong \pi_n(\Omega B\mathcal{Q}) \qquad n \geq 0$$

Explanation

- $B\mathcal{Q}$ is classifying space of a category
- $\Omega X = \mathrm{Map}(S^1, X)$ for any X

  ($\pi_n(\Omega X) \cong \pi_{n+1}(X)$ for $n \geq 0$)
- $\Omega B\mathcal{Q}$ is a topological group-completion
  - group-completion Gr on π_0
  - isomorphism on homology when π_0 inverted

The $\mathcal{Q}$-construction is purely algebraic; doesn't rely on plus construction. Iterated $\mathcal{Q}$-construction (Waldhausen's $S_\bullet$) has further structure.

Quillen's higher K-groups: + = $\mathbb{Q}$

Theorem (Quillen)

$$K_0(R) \times BGL(R)^+ \simeq \Omega B\mathbb{Q}.$$

This gives two completely different ways to approach algebraic K-theory. Each version has both conceptual and calculational advantages.

Proof sketch: $+ = S^{-1}S = \mathbb{Q}$

Let $S = \mathrm{Proj}^{\mathrm{iso}}_{f.g.}(R)$. Define localization of categories, $S^{-1}S$, and prove

$$K_0(R) \times BGL(R)^+ \simeq BS^{-1}S \simeq \Omega B\mathbb{Q}.$$

(We will say more about $S^{-1}S$, but not $\mathbb{Q}$.)

$S^{-1}S$

Let $S = (S, \otimes)$ be a symmetric monoidal category with

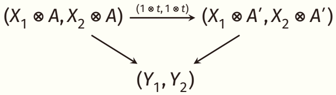
- every morphism invertible, (S is a groupoid)
- faithful translations. ($X \otimes A \cong Y \otimes A$ iff $X \cong Y$)

Define new category $S^{-1}S$ with formal inverses to $\otimes$.

- Objects (X_1, X_2) pairs of objects from S.
  (formal fractions)
- Morphisms $(X_1, X_2) \to (Y_1, Y_2)$ equivalence classes of triples (A, f_1, f_2): (cancellation morphisms)

$$(f_1, f_2) : (X_1 \otimes A, X_2 \otimes A) \to (Y_1, Y_2).$$

- Equivalence: (morphisms of cancellation morphisms)
  $(A, f_1, f_2) \sim (A', f_1', f_2')$ if
  $\exists t : A \to A'$ such that



$S^{-1}S$

Examples

- $S = \mathrm{Proj}^{iso}_{f.g.}(R)$ with $\oplus$
- $S = \coprod_n \Sigma_n \simeq \mathrm{FinSet}$ with (block) sum / disjoint union.

Theorem (Quillen, Grayson)

The inclusion $S \to S^{-1}S$ induces a topological group-completion. That is:

$$BS \to BS^{-1}S,$$

- induces group-completion on π_0;
- is a homology localization:

$$H_*(BS) \longrightarrow H_*(BS^{-1}S) \cong H_*(BS)[\pi_0(BS)^{-1}].$$

$S^{-1}S$

Generalization of $GL(R)$: Given a sequence of objects and inclusions

$$\cdots \hookrightarrow \mathrm{Aut}_S(A_n) \hookrightarrow \mathrm{Aut}_S(A_{n+1}) \hookrightarrow \cdots,$$

we have a group $\mathrm{Aut}(S) = \mathrm{colim}_n \mathrm{Aut}_S(A_n)$.
(This is deceptively simple.)

If (as in our examples), the sequence is cofinal in S, one proves

$$BS \to K_0(S) \times B\mathrm{Aut}(S)$$

is an isomorphism on localized homology $H_*(\;\_\_\;)[\pi_0(BS)^{-1}]$.
(This is another topological group-completion.)

$S^{-1}S$

Therefore the solid arrows are isomorphisms on $H_*(\,\_\_\,)[\pi_0(BS)^{-1}]$

$$\begin{array}{ccc} BS & \longrightarrow & K_0(S) \times B\mathrm{Aut}(S) \\ \downarrow & & \downarrow \\ BS^{-1}S & \dashrightarrow & K_0(S) \times B\mathrm{Aut}(S)^+ \end{array}$$

By universal property we have the dashed arrow inducing a homology isomorphism. Therefore a homotopy equivalence (of simple spaces)

$$BS^{-1}S \simeq K_0(S) \times B\mathrm{Aut}(S)^+.$$

This is half of the $+ = \mathrm{Q}$ theorem.

Example: $K_2(R) \cong H_2(E(R))$

We have two homotopy fiber sequences:

$$\begin{array}{ccccc} BE & \longrightarrow & B\mathrm{Aut}(S) & \longrightarrow & P_1\, BS^{-1}S_0 \\ \downarrow & & \downarrow & & \| \\ F & \longrightarrow & B\mathrm{Aut}(S)^+ & \longrightarrow & P_1\, BS^{-1}S_0 \end{array}$$

- $BS^{-1}S_0$ denotes basepoint component
- P_1 denotes first Postnikov truncation
- $E = [\mathrm{Aut}(S), \mathrm{Aut}(S)]$ (Note top is homotopy fiber sequence)
- F is defined to be homotopy fiber on the bottom row
  $\pi_1(F) = 0;\ \pi_2(F) \cong \pi_2(B\mathrm{Aut}(S)^+)$
- Vertical maps are homology isomorphisms (by SSS)

$$H_2(E) \cong H_2(BE) \cong H_2(F) \cong \pi_2(F) \cong \pi_2(B\mathrm{Aut}(S)^+) = K_2(S)$$

Recap of background

- Algebraic K groups of a ring connect geometric and number-theoretic information.
  - $K_0(R) = Gr(\text{Proj}_{\text{f.g.}}(R))$
  - $K_1(R) = H_1(GL(R)) = GL(R)^{\text{ab}} = GL(R)/E(R)$
  - $K_2(R) = H_2(E(R))$
  - $K_n(R) = \pi_n\left(BGL(R)^+\right)$.
- Quillen's "$+ = S^{-1}S = \mathbf{Q}$"
  - Different but equivalent defintions of higher K groups give different calculational and conceptual information.
  - Bridge, $S^{-1}S$, is a categorical construction whose classifying space realizes both $+$ and $\mathbf{Q}$ constructions.
  - Both homotopical and algebraic tools inform computation of $K_*(R)$.

Motivations for higher-categorical algebra

Who could ask for anything more?!

- Relative K-theory: Given a map of rings $f : R \to T$,

$$K_n(f) = \pi_n \mathrm{hofib}\big(BGL(R)^+ \to BGL(T)^+ \big)$$

- Hermitian K-theory: If R is a ring with involution (e.g. complex conjugation), Hermitian K-theory is defined via homotopy fixed points of $BGL(R)^+$.
  - Relates to topology of manifolds (e.g. diffeomorphism classes)
  - Relates to motivic homotopy theory

- Use Postnikov P_2 to make an algebraic calculation of $K_3(R)$? (Recall $K_3(\mathbb{Z}) = \mathbb{Z}/48 \twoheadrightarrow \pi_3^s = \mathbb{Z}/24$.)

Are there categories whose classifying spaces compute these?!

Symmetric monoidal algebra in dimension 2

Symmetric monoidal 2-categories are like symmetric monoidal categories with another level of structure.

(product on objects, morphisms, and 2-morphisms)

(we mean a 2-category that is symmetric monoidal as a *bicategory*)

Example *Bimod* has objects which are rings; $Bimod(R, T)$ is the category of (R, T)-bimodules.

- Tensor product provides "composition" of $R \xrightarrow{M} T \xrightarrow{N} V$.
- Tensor product of rings provides symmetric monoidal structure.

Example *Cat* has objects which are categories; $Cat(C, D)$ is the category of functors and natural transformations.

- Cartesian product of categories provides a symmetric monoidal structure.

Symmetric monoidal algebra in dimension 2

Better examples:

- relative constructions
- fixed point constructions
- telescope (colimit) constructions

$S^{-1}S$ for 2-categories

Let S be a symmetric monoidal 2-category and suppose:

- all morphisms and 2-morphsisms are invertible
  (*S* is a 2-groupoid)
- S has faithful translations

Theorem (Gurski-J.-Osorno)

There is a symmetric monoidal 2-category $S^{-1}S$ with

$$S \longrightarrow S^{-1}S$$

inducing a topological group-completion.

- group-completion on π_0
- isomorphism on localized homology

$S^{-1}S$ for 2-categories

Sketch proof

- $S^{-1}S$: same idea as 1-categorical case, but include 2-morphisms instead of equivalence relation on morphisms.
- Use 2-categorical comma construction to analyze fibers. (This took a while to figure out.)
- Homology spectral sequence collapses after inverting $\pi_0(BS)$. (Just like 1-categorical case.)

$+ = S^{-1}S$ for 2-categories

Given a sequence of objects and faithful functors

$$\cdots \hookrightarrow \mathrm{Aut}_S(A_n) \hookrightarrow \mathrm{Aut}_S(A_{n+1}) \hookrightarrow \cdots,$$

we have a categorical group $\mathrm{Aut}(S) = \mathrm{colim}_n \mathrm{Aut}_S(A_n)$.

(monoidal category with objects and morphisms are invertible)

If (as in examples), the sequence is cofinal, we have:

Theorem (80% done; Fontes-Gurski-J.)

$$BS^{-1}S \simeq K_0(S) \times B\mathrm{Aut}(S)^+$$

Key step Compare colimit of 1-object 2-categories, v.s. 1-object 2-category of colimit (Σ means 1-object "suspension")

$$\mathrm{colim}_n \Sigma \mathrm{Aut}_S(A_n) \quad \text{v.s.} \quad \Sigma \mathrm{colim}_n \mathrm{Aut}_S(A_n)$$

(1-categorical case completely straightforward)

Application to K_3

Theorem (In progress; J.-Parab)

There is a commutator subcategory E such that the following is a homotopy fiber sequence.

$$BE \longrightarrow B\mathrm{Aut}(S) \longrightarrow P_2\, BS^{-1}S_0$$

Corollary

$$K_3(R) \cong H_3(BE)$$

(same method as K_2 calculation using P_1)

Conjecture (Fontes)

The commutator category E is a categorification of the Steinberg group.

- $St(R)$ is the universal central extension of the commutator subgroup of $GL(R)$.
- Gersten (1973) proved $K_3(R) \cong H_3(St(R))$ using very different methods.

Conclusion

Algebraic K-theory for 2-categories
joint with
E. Fontes, N. Gurski, A.M. Osorno, and A. Parab

- Background on algebraic K-theory
  - K_0, K_1, K_2
  - $BGL(R)^+$
  - $+ = S^{-1}S = \mathbf{Q}$
- Sketch of 2-categorical K-theory
  - Motivations: K_3, Hermitian, relative
  - Current issues: colimits of monoidal 1-categories
  - Future plans: understanding the Steinberg group

Thank You!