

Algebraic K -theory for 2-categories

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joint work with E. Fontes, N. Gurski, and A. Osorno

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Abstract

Quillen recognized the higher algebraic K -groups of a ring R as homotopy groups of a certain topological space, $BGL(R)^+$. We review some of the basic definitions and computations via categorical algebra. We then describe how a 2-categorical extension of this theory leads to a new model for $K_3(R)$, together with more general applications. We will give a mild sampling of key technical details and close with some of the problems we're currently working on. The work we present is joint with Gurski-Osorno, Fontes, and Fontes-Gurski.

Outline

- ▶ Algebraic K -groups of a ring
 - ▶ Low-dimensional cases
 - ▶ Quillen's higher K -groups
 - ▶ Application for K_2
- ▶ 2-categorical analogue
 - ▶ Examples of symmetric monoidal 2-categories
 - ▶ Generalizations of Quillen's constructions
 - ▶ Application for K_3

Algebraic K -groups of a ring: $K_0(R)$

Ring R

$\text{Proj}_{f.g.}(R)$ = finitely-generated projective R -modules
(symmetric monoidal category under \otimes .)

Definition

$$K_0(R) = \text{Gr}(\text{Proj}_{f.g.}(R) / \cong).$$

(Gr = group-completion; known as Grothendieck construction)

Example for field F , $K_0(F) = \mathbb{Z}$.

Example Application $\det: K_0(R) \rightarrow \text{Pic}(R)$
is a surjection.

Algebraic K -groups of a ring: $K_0(R)$

Generalize to Grothendieck group of other symmetric monoidal categories S :

Definition

$$K_0(S) = Gr(S / \cong).$$

Examples:

- ▶ $\text{Proj}_{f.g.}(R)$
- ▶ vector bundles on topological spaces
- ▶ representations of finite groups

Algebraic K -groups of a ring: $K_1(R)$

Let $GL(R)$ be the infinite general linear group

$$GL(R) = \operatorname{colim}(\dots \hookrightarrow GL_n(R) \hookrightarrow GL_{n+1}(R) \hookrightarrow \dots).$$

(union of groups)

Definition

$$K_1(R) = GL(R)^{\text{ab}} = GL(R) / [GL(R), GL(R)].$$

Equivalently,

$$K_1(R) = H_1(GL(R)).$$

Algebraic K -groups of a ring: $K_1(R)$

Theorem (Localization exact sequence)

Let S be a multiplicatively closed set of central elements in R . Then there is an exact sequence

$$K_1(R) \rightarrow K_1(S^{-1}R) \rightarrow K_0(\text{Ft}_S(R)) \rightarrow K_0(R) \rightarrow K_0(S^{-1}R).$$

($\text{Ft}_S(R)$ = S -torsion R -modules with finite-length projective resolutions)

Theorem (Fundamental Theorem)

There is an exact sequence

$$0 \rightarrow K_1(R) \rightarrow K_1(R[t]) \oplus K_1(R[t^{-1}]) \rightarrow K_1(R[t, t^{-1}]) \rightarrow K_0(R) \rightarrow 0.$$

Algebraic K -groups of a ring: $K_2(R)$

Lemma (Whitehead)

$$[GL(R), GL(R)] = E(R) = \operatorname{colim}_n E_n(R)$$

where $E_n(R)$ is the **elementary group** generated by elementary $n \times n$ matrices.

(ones on diagonal; single off-diagonal entry)

Equivalent definitions

$$K_1(R) = GL(R)^{\text{ab}} = GL(R)/E(R) = H^1(GL(R))$$

Algebraic K -groups of a ring: $K_2(R)$

The **Steinberg group**, $St(R)$ is generated by formal symbols $x_{ij}(r)$ for $r \in R$, subject to elementary relations.
(products and commutators of elementary matrices)

Note: $E(R)$ generally has more relations than $St(R)$.

Definition

$$K_2(R) = \ker(St(R) \rightarrow E(R)).$$

Theorem (Steinberg)

$K_2(R)$ is the center of $St(R)$. In particular, $K_2(R)$ is abelian.

Theorem (Bass)

$$K_2(R) \cong H_2(E(R)).$$

Properties of low-dimensional K -groups

Looking at the definitions, it may be unclear that the groups $K_i(R)$ are part of any reasonable sequence. Here are some clues.

- ▶ Localization exact sequence
- ▶ Fundamental Theorems for $R[t, t^{-1}]$
- ▶ $K_1(R)$ and $K_2(R)$ are modules over $K_0(R)$
- ▶ Product $K_1(R) \otimes K_1(R) \rightarrow K_2(R)$.

Quillen's higher K -groups

Definition

$$K_n(R) = \pi_n(BGL(R)^+), \quad n > 0$$

Explanation (Part 1) BG is the **classifying space** of a group G . (we will take $G = GL(R)$)

BG is the base of a principal G -bundle

$$G \rightarrow EG \rightarrow BG$$

- ▶ EG contractible and has free G -action
- ▶ therefore $BG \simeq K(G, 1)$, an Eilenberg-Mac Lane space

Example: $B\mathbb{Z} = S^1$

$$\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1$$

Quillen's higher K -groups

Definition

$$K_n(R) = \pi_n(BGL(R)^+), \quad n > 0$$

Explanation (Part 2) The plus construction X^+ on a topological space X has two properties:

- ▶ $\pi_1(X^+) = \pi_1(X)^{\text{ab}} \cong H_1(X)$
- ▶ a map $X \rightarrow X^+$ inducing a homology isomorphism

Thus we certainly have

$$\pi_1(BGL(R)^+) \cong GL(R)^{\text{ab}} = K_1(R).$$

Quillen's higher K -groups

Definition

$$K_n(R) = \pi_n(BGL(R)^+), \quad n > 0$$

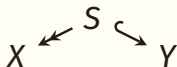
Explanation (Part 3) Why is this definition such a good one? (partial answer)

- ▶ Extends localization exact sequence and fundamental theorem.
- ▶ Explains graded ring structure on $K_n(R)$.
- ▶ Explains connection between algebraic K -groups and homotopy theory.

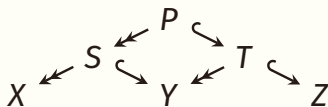
Quillen's higher K -groups: Second version

Let $\mathcal{Q} = \mathcal{Q}(\text{Proj}_{f.g.}(R))$ be the following category.

- ▶ Objects are those of $\text{Proj}_{f.g.}(R)$
- ▶ Morphisms $X \rightarrow Y$ are injection/surjection spans



- ▶ Composition is via pullback



Quillen's higher K -groups: Second version

Definition (Alternate)

$$K_n(R) = \pi_{n+1}(BQ) \cong \pi_n(\Omega BQ) \quad n \geq 0$$

Explanation

- ▶ BQ is classifying space of a category
- ▶ $\Omega X = \text{Map}(S^1, X)$ for any X
($\pi_n(\Omega X) \cong \pi_{n+1}(X)$ for $n \geq 0$)
- ▶ ΩBQ is a topological group-completion
 - ▶ group-completion Gr on π_0
 - ▶ isomorphism on homology when π_0 inverted

The Q -construction is purely algebraic; doesn't rely on plus construction. Iterated Q -construction (Waldhausen's S_\bullet) has further structure.

Quillen's higher K -groups: $+$ = Q

Theorem (Quillen)

$$K_0(R) \times BGL(R)^+ \simeq \Omega BQ.$$

This gives two completely different ways to approach algebraic K -theory. Each version has both conceptual and calculational advantages.

Proof sketch: $+$ = $S^{-1}S$ = Q

Let $S = \text{Proj}_{f.g.}^{iso}(R)$. Define localization of categories, $S^{-1}S$, and prove

$$K_0(R) \times BGL(R)^+ \simeq BS^{-1}S \simeq \Omega BQ.$$

(We will say more about $S^{-1}S$, but not Q .)

$S^{-1}S$

Let $S = (S, \otimes)$ be a symmetric monoidal category with

- ▶ every morphism invertible, (S is a groupoid)
- ▶ faithful translations. ($X \otimes A \cong Y \otimes A$ iff $X \cong Y$)

Define new category $S^{-1}S$ with formal inverses to \otimes .

- ▶ Objects (X_1, X_2) pairs of objects from S .
(formal fractions)
- ▶ Morphisms $(X_1, X_2) \rightarrow (Y_1, Y_2)$ equivalence classes of triples (A, f_1, f_2) : (cancellation morphisms)

$$(f_1, f_2): (X_1 \otimes A, X_2 \otimes A) \rightarrow (Y_1, Y_2).$$

- ▶ Equivalence: (morphisms of cancellation morphisms)

$(A, f_1, f_2) \sim (A', f'_1, f'_2)$ if
 $\exists t: A \rightarrow A'$ such that

$$\begin{array}{ccc} (X_1 \otimes A, X_2 \otimes A) & \xrightarrow{(1 \circ t, 1 \circ t)} & (X_1 \otimes A', X_2 \otimes A') \\ & \searrow & \swarrow \\ & (Y_1, Y_2) & \end{array}$$

$S^{-1}S$

Examples

- ▶ $S = \text{Proj}_{f.g.}^{\text{iso}}(R)$ with \oplus
- ▶ $S = \coprod_n \Sigma_n \simeq \text{FinSet}$ with (block) sum / disjoint union.

Theorem (Quillen, Grayson)

The inclusion $S \rightarrow S^{-1}S$ induces a topological group-completion. That is:

$$BS \rightarrow BS^{-1}S,$$

- ▶ induces group-completion on π_0 ;
- ▶ is a homology localization:

$$H_*(BS) \longrightarrow H_*(BS^{-1}S) \cong H_*(BS)[\pi_0(BS)^{-1}].$$

$S^{-1}S$

Generalization of $GL(R)$: Given a sequence of objects and inclusions

$$\dots \hookrightarrow \text{Aut}_S(A_n) \hookrightarrow \text{Aut}_S(A_{n+1}) \hookrightarrow \dots,$$

we have a **group** $\text{Aut}(S) = \text{colim}_n \text{Aut}_S(A_n)$.
(This is deceptively simple.)

If (as in our examples), the sequence is cofinal in S , one proves

$$BS \rightarrow K_0(S) \times B\text{Aut}(S)$$

is an isomorphism on localized homology
 $H_*(\text{---})[\pi_0(BS)^{-1}]$.

(This is another topological group-completion.)

$S^{-1}S$

Therefore the solid arrows are isomorphisms on $H_*(\text{---})[\pi_0(BS)^{-1}]$

$$\begin{array}{ccc} BS & \longrightarrow & K_0(S) \times B\text{Aut}(S) \\ \downarrow & & \downarrow \\ BS^{-1}S & \dashrightarrow & K_0(S) \times B\text{Aut}(S)^+ \end{array}$$

By universal property we have the dashed arrow inducing a **homology isomorphism**. Therefore a **homotopy equivalence** (of simple spaces)

$$BS^{-1}S \simeq K_0(S) \times B\text{Aut}(S)^+.$$

This is half of the $+ = Q$ theorem.

Example: $K_2(R) \cong H_2(E(R))$

We have two homotopy fiber sequences:

$$\begin{array}{ccccc} BE & \longrightarrow & B\text{Aut}(S) & \longrightarrow & P_1 BS^{-1}S_0 \\ \downarrow & & \downarrow & & \parallel \\ F & \longrightarrow & B\text{Aut}(S)^+ & \longrightarrow & P_1 BS^{-1}S_0 \end{array}$$

- ▶ $BS^{-1}S_0$ denotes basepoint component
- ▶ P_1 denotes first Postnikov truncation
- ▶ $E = [\text{Aut}(S), \text{Aut}(S)]$ (Note top is homotopy fiber sequence)
- ▶ F is defined to be homotopy fiber on the bottom row
 $\pi_1(F) = 0; \pi_2(F) \cong \pi_2(B\text{Aut}(S)^+)$
- ▶ Vertical maps are homology isomorphisms (by SSS)

$$H_2(E) \cong H_2(BE) \cong H_2(F) \cong \pi_2(F) \cong \pi_2(B\text{Aut}(S)^+) = K_2(S)$$

Recap of background

- ▶ Algebraic K groups of a ring connect geometric and number-theoretic information.
 - ▶ $K_0(R) = Gr(\text{Proj}_{f.g.}(R))$
 - ▶ $K_1(R) = H_1(GL(R)) = GL(R)^{ab} = GL(R)/E(R)$
 - ▶ $K_2(R) = H_2(E(R))$
 - ▶ $K_n(R) = \pi_n(BGL(R)^+)$.
- ▶ Quillen's “ $+$ = $S^{-1}S = Q$ ”
 - ▶ Different but equivalent definitions of higher K groups give different calculational and conceptual information.
 - ▶ Bridge, $S^{-1}S$, is a categorical construction whose classifying space realizes both $+$ and Q constructions.
 - ▶ Both homotopical and algebraic tools inform computation of $K_*(R)$.

Motivations for higher-categorical algebra

Who could ask for anything more?!

- ▶ Relative K -theory: Given a map of rings $f: R \rightarrow T$,

$$K_n(f) = \pi_n \text{hofib}(BGL(R)^+ \rightarrow BGL(T)^+)$$

- ▶ Hermitian K -theory: If R is a ring with involution (e.g. complex conjugation), Hermitian K -theory is defined via homotopy fixed points of $BGL(R)^+$.
 - ▶ Relates to topology of manifolds (e.g. diffeomorphism classes)
 - ▶ Relates to motivic homotopy theory
- ▶ Use Postnikov P_2 to make an algebraic calculation of $K_3(R)$? (Recall $K_3(\mathbb{Z}) = \mathbb{Z}/48 \rightarrow \pi_3^S = \mathbb{Z}/24$.)

Are there categories whose classifying spaces compute these?!

Symmetric monoidal algebra in dimension 2

Symmetric monoidal 2-categories are like symmetric monoidal categories with another level of structure.

(product on objects, morphisms, and 2-morphisms)

Example *Bimod* has objects which are rings; *Bimod*(R, T) is the category of (R, T)-bimodules.

- ▶ Tensor product provides “composition” of $R \xrightarrow{M} T \xrightarrow{N} V$.
- ▶ Tensor product of rings provides symmetric monoidal structure.

Example *Cat* has objects which are categories; *Cat*(C, D) is the category of functors and natural transformations.

- ▶ Cartesian product of categories provides a symmetric monoidal structure.

Symmetric monoidal algebra in dimension 2

Better examples:

- ▶ relative constructions
- ▶ fixed point constructions
- ▶ telescope (colimit) constructions

$S^{-1}S$ for 2-categories

Let S be a symmetric monoidal 2-category and suppose:

- ▶ all morphisms and 2-morphisms are invertible
(S is a 2-groupoid)
- ▶ S has faithful translations

Theorem (Gurski-J.-Osorno)

There is a symmetric monoidal 2-category $S^{-1}S$ with

$$S \longrightarrow S^{-1}S$$

inducing a topological group-completion.

- ▶ group-completion on π_0
- ▶ isomorphism on localized homology

$S^{-1}S$ for 2-categories

Sketch proof

- ▶ $S^{-1}S$: same idea as 1-categorical case, but include 2-morphisms instead of equivalence relation on morphisms.
- ▶ Use 2-categorical comma construction to analyze fibers. (This took a while to figure out.)
- ▶ Homology spectral sequence collapses after inverting $\pi_0(BS)$. (Just like 1-categorical case.)

$+ = S^{-1}S$ for 2-categories

Given a sequence of objects and faithful functors

$$\dots \hookrightarrow \text{Aut}_S(A_n) \hookrightarrow \text{Aut}_S(A_{n+1}) \hookrightarrow \dots,$$

we have a **categorical group** $\text{Aut}(S) = \text{colim}_n \text{Aut}_S(A_n)$.

(monoidal category with objects and morphisms are invertible)

If (as in examples), the sequence is cofinal, we have:

Theorem (90% done; Fontes-Gurski-J.)

$$BS^{-1}S \simeq K_0(S) \times B\text{Aut}(S)^+$$

Key step Compare colimit of 1-object 2-categories, v.s.
1-object 2-category of colimit (Σ means 1-object “suspension”)

$$\text{colim}_n \Sigma \text{Aut}_S(A_n) \quad \text{v.s.} \quad \Sigma \text{colim}_n \text{Aut}_S(A_n)$$

(1-categorical case completely straightforward)

Application to K_3

Theorem (In progress; Fontes-J.)

There is a commutator subcategory E such that the following is a homotopy fiber sequence.

$$BE \longrightarrow B\text{Aut}(S) \longrightarrow P_2 BS^{-1}S_0$$

Corollary

$$K_3(R) \cong H_3(BE) \quad (\text{same method as } K_2 \text{ calculation using } P_1)$$

Conjecture

The commutator category E is a categorification of the Steinberg group.

- ▶ $St(R)$ is the universal central extension of the commutator subgroup of $GL(R)$.
- ▶ Gersten (1973) proved $K_3(R) \cong H_3(St(R))$ using very different methods.

Conclusion

Algebraic K -theory for 2-categories

joint with E. Fontes, N. Gurski, and A. Osorno

- ▶ Background on algebraic K -theory
 - ▶ K_0, K_1, K_2
 - ▶ $BGL(R)^+$
 - ▶ $+ = S^{-1}S = Q$
- ▶ Sketch of 2-categorical K -theory
 - ▶ Motivations: K_3 , Hermitian, relative
 - ▶ Current issues: colimits of monoidal 1-categories
 - ▶ Future plans: understanding the Steinberg group

Thank You!