Algebra Morphism Coherence

OR

Universal pseudomorphisms, with applications to diagrammatic coherence for braided and symmetric monoidal functors

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joint work with N. Gurski

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Abstract (outline for the talk)

Goal: Explain the baroque title

- This talk introduces coherence results for structure-preserving functors.
- We begin with motivating examples for braided and symmetric monoidal functors.
- Then, we explain how the coherence theorems for monoidal categories (plain, braided, and symmetric) follow from characterizations of free algebras over a 2-monad.
- Our coherence for algebra morphisms uses this same approach, via a theory of *universal pseudomorphisms*.

Based on joint work with Nick Gurski.

Example 1

Braided strong monoidal $f : (A, +, \beta) \to (A', \cdot, \beta)$

Diagram:

$$
\begin{array}{ccccc}
f(a) \cdot f(a) \cdot f(a) & \xrightarrow{f_2 \cdot 1} & f(a+a) \cdot f(a) & \xrightarrow{\beta} & f(a) \cdot f(a+a) \\
{\scriptstyle f_2}\downarrow & & & & \downarrow{\scriptstyle f_2} \\
f(a+a+a) & \xrightarrow{f(1+\beta)} & f(a+a+a) & \xrightarrow{f(\beta+1)} & f(a+a+a)
\end{array}
$$

Dissolution: (treat f_2 as identity!)

$$
\begin{array}{ccccc}
\big(f(a), f(a), f(a) \big) & \xrightarrow{1} & \big(f(a), f(a), f(a) \big) & \xrightarrow{\beta_{(1\ 2\ 3)}} & \big(f(a), f(a), f(a) \big) \\
{\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\
\big(f(a), f(a), f(a) \big) & \xrightarrow{(1, \beta)} & \big(f(a), f(a), f(a) \big) & \xrightarrow{(\beta, 1)} & \big(f(a), f(a), f(a) \big)
\end{array}
$$

- The dissolution diagram looks simpler!
- The dissolution diagram looks *completely different*!

Example Discussion

Two weird and surprising things:

1. The monoidal constraints of f could have nontrivial braidings.
   Replacing constraints with identities sounds like forgetting nontrivial data. It is!

2. The monoidal constraints of f generally have domain/codomain that are *NOT equal*. So, there *is not* an identity morphism between them; we also have to swap out objects.
   That sounds complicated. It isn't!

Example 2

Braided strong monoidal $f : (A, +, \beta) \to (A', \cdot, \beta)$

Diagram:

$$\begin{array}{ccc}
f(a) \cdot f(b) \cdot f(c) \cdot f(d) & \xrightarrow{f_2 \cdot f_2} & f(a + b) \cdot f(c + d) \\
\downarrow {\scriptstyle 1 \cdot \beta \cdot 1} & & \downarrow {\scriptstyle f_2} \\
f(a) \cdot f(c) \cdot f(b) \cdot f(d) & & f(a + b + c + d) \\
\downarrow {\scriptstyle f_2 \cdot f_2} & & \downarrow {\scriptstyle f(1 + \beta + 1)} \\
f(a + c) \cdot f(b + d) & \xrightarrow{f_2} & f(a + c + b + d)
\end{array}$$

This is the diagram to verify whether the natural transformation f_2 is *monoidal* natural.

Dissolve the diagram: recognize formal sums/products and applications of f.

Example 2 Dissolution

Dissolution:

$$\begin{array}{ccc}
\left(f(a)\,,\ f(b)\,,\ f(c)\,,\ f(d) \right) & \xrightarrow{\quad 1 \quad} & \left(f(a)\,,\ f(b)\,,\ f(c)\,,\ f(d) \right) \\
{\scriptstyle (1,\beta,1)}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\
\left(f(a)\,,\ f(c)\,,\ f(b)\,,\ f(d) \right) & & \left(f(a)\,,\ f(b)\,,\ f(c)\,,\ f(d) \right) \\
{\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle (1,\beta,1)} \\
\left(f(a)\,,\ f(c)\,,\ f(b)\,,\ f(d) \right) & \xrightarrow{\quad 1 \quad} & \left(f(a)\,,\ f(c)\,,\ f(b)\,,\ f(d) \right)
\end{array}$$

This diagram commutes.

Theorem: therefore original also commutes.

Note: yes, these examples are also easy to check directly

Main Application: coherence theory for general diagrams involving strong monoidal f

Coherence for monoidal categories

Let's review coherence for
plain/symmetric/braided monoidal categories

Diagrammatic Coherence: Does the diagram commute?

Note: Diagram in a plain/symmetric/braided monoidal category;
no functor involved yet

Coherence for monoidal categories

(Diagrammatic Coherence)

Plain Monoidal [ML98]. Every formal diagram commutes.

Equivalently: Every parallel pair of formal morphisms are equal.

Symmetric Monoidal [ML98]. Two parallel formal morphisms are equal if they have the same underlying permutation.

Braided Monoidal [JS93]. Two parallel formal morphisms are equal if they have the same underlying braid.

What is a *formal* diagram!?

Coherence: Formal diagrams

Basic Idea: Consists only of structure morphisms
Doesn’t use “accidental” relations

Non-Examples: Joyal-Street monoidal structures via group cocycles.
Many nontrivial diagrams of structure morphisms

More Precise Idea: Formal diagrams come from a *free* monoidal category (plain/symmetric/braided).

Coherence: Free algebras

Slogan to be explained

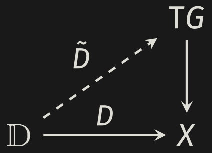
Coherence is when you characterize a free algebra. The more you characterize the free algebra, the more coherence you have.

Definition. A *diagram* in a plain/symmetric/braided monoidal category X is a functor

$$D\colon \mathbb{D} \to X$$

for a small category $\mathbb{D}$.

A diagram $(\mathbb{D}, D)$ is *formal* if it lifts to a free plain/symmetric/braided monoidal category on a set $G \subset \mathrm{ob}X$.
(*G* for generators)



Coherence: Free algebras

T = M/S/B in any of the three free/forgetful adjunctions:

$$\mathcal{C}at \underset{\mathsf{u}}{\overset{\mathsf{M}}{\rightleftarrows}} \mathcal{M}on\mathcal{C}at\,,\quad \mathcal{C}at \underset{\mathsf{u}}{\overset{\mathsf{S}}{\rightleftarrows}} \mathcal{S}ym\mathcal{M}on\mathcal{C}at\,,\quad \mathcal{C}at \underset{\mathsf{u}}{\overset{\mathsf{B}}{\rightleftarrows}} \mathcal{B}r\mathcal{M}on\mathcal{C}at$$

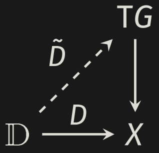
Free algebras on a set of objects G [ML98, JS93]:

- $\mathsf{M}G$ is equivalent to $\overline{\mathsf{M}}G$: strict monoidal, objects are lists, and morphisms are all identities.
- $\mathsf{S}G$ is equivalent to $\overline{\mathsf{S}}G$: strict monoidal, objects are lists, and morphisms are permutations.
- $\mathsf{B}G$ is equivalent to $\overline{\mathsf{B}}G$: strict monoidal, objects are lists, and morphisms are braidings.

Characterization of free morphisms implies diagrammatic coherence

Coherence: Free algebras

Suppose $(\mathbb{D}, D)$ a formal diagram in X with lift $(\mathbb{D}, \tilde{D})$ to $\mathrm{T}G$ (for $G \subset \mathrm{ob}X$).
Key: If $(\mathbb{D}, \tilde{D})$ commutes in $\mathrm{T}G$, then the original diagram $(\mathbb{D}, D)$ in X also commutes.



Slogan (again)

Coherence is when you characterize a free algebra. The more you characterize the free algebra, the more coherence you have.

This general approach works for any algebraic structure encoded by a (2-)monad (free/forgetful adjunction).

- Structures defined by data and axioms
- Could be a 2-monad on $\mathcal{C}at$, or more general $\mathcal{K}$
- Motivates significant interest in 2-monad theory
- Leads to more general and abstract coherence

Pseudomorphism Coherence

What about diagrammatic coherence involving pseudomorphisms?

Definition. A T-*pseudomorphism* between T-algebras is a structure-preserving morphism

$$f : X \rightsquigarrow X'$$

(zigzag arrow = pseudo strength)
(pseudo = up to isomorphism)

Examples. Plain/Symmetric/Braided strong monoidal functors (f, f_2, f_0)

Pseudomorphism Coherence

Question

Suppose we have a coherence theory for T-algebras X and X'. (i.e., characterization of free algebras)
How can we tell when formal diagrams involving data of a pseudomorphism f commute?

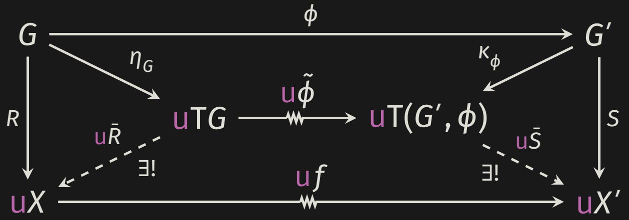
Call our answer:
Diagrammatic Coherence for Pseudomorphisms

(Note: only *pseudo*morphisms; not lax morphisms)
(see last slide for non-example in lax case)

Pseudomorphism Coherence: (UPC)

Suppose given T-algebras X and X' and morphism $\phi : G \to G'$ in underlying 2-category $\mathcal{K}(= \mathcal{C}at)$.
(In applications: $\phi = f_{\text{ob}}$.)

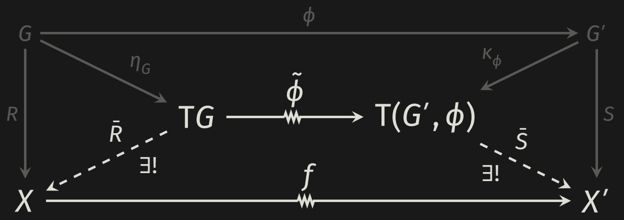
A *universal pseudomorphism construction (UPC)* for ϕ is a T-pseudomorphism $\tilde{\phi} : \mathrm{T}G \to \mathrm{T}(G', \phi)$ such that:



Given f, R, S, there are unique $\bar{R}$ and $\bar{S}$
Equivalently: a certain adjunction of arrow categories
What does this mean!?

Pseudomorphism Coherence: (UPC)

In $\mathrm{T}\mathcal{Alg}_{ps}$: (2-category with T-pseudomorphisms)



- $\bar{R}$ and $\bar{S}$ strict T-morphisms induced on generators
- Restricting to G: $f|_G = \phi|_G = \tilde{\phi}|_G$
- $\mathrm{T}G$ is freely generated by $x \in G$
- $\mathrm{T}(G', \phi)$ is freely generated by: $x' \in G'$, $\phi[w]$ for $w \in \mathrm{T}G$, and formal constraint morphisms
- Taking $R = \eta_G$ and $S = \eta_{G'}$ gives canonical strict $\Delta = \bar{\eta} : \mathrm{T}(G', \phi) \to \mathrm{T}G'$

Pseudomorphism Coherence Theorem

Main Theorem [GJ23]

Suppose T is one of M, S, B, or *many* other 2-monads.
(finitary on bicomplete domain is sufficient, not necessary)

Then T admits a *UPC* $\tilde{\phi}\colon \mathrm{T}G \rightsquigarrow \mathrm{T}(G', \phi)$
such that $\Delta\colon \mathrm{T}(G', \phi) \to \mathrm{T}G'$
is an *equivalence* of T-algebras.

Proof Remark. The conditions for T are often equivalent to T admitting a *pseudomorphism classifier*:

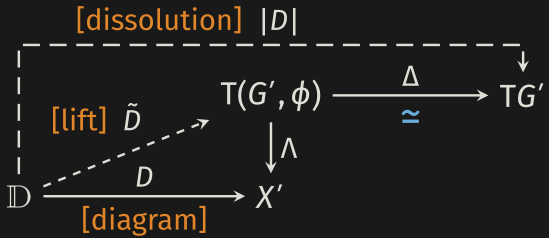
$$\mathrm{T}\mathcal{A}lg_{ps} \underset{}{\overset{\mathrm{Q}}{\rightleftarrows}} \mathrm{T}\mathcal{A}lg_{str}$$

(2-adjunction between pseudo- and strict morphism variants)
(recall mention of more abstract 2-monadic coherence)

Diagrammatic Pseudomorphism Coherence

$f : X \rightsquigarrow X'$ is a T-pseudomorphism;
G and G' are object sets; let $\phi = f_{\mathrm{ob}}$
Taking $R = 1_G$ and $S = 1_{G'}$ gives universal $\Lambda = \bar{1}$

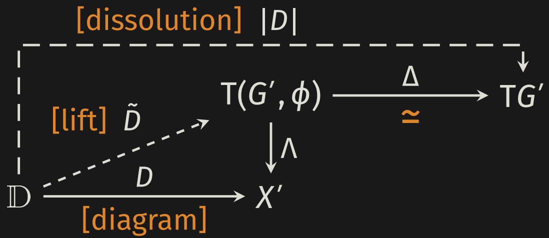
Definition. *formal diagram for f* and *dissolution*:



Theorem. Δ is an equivalence.

Corollary. Suppose $(\mathbb{D}, D)$ is a formal diagram with lift $\tilde{D}$.
If the *dissolution* $|D| = \Delta\,\tilde{D}$ commutes, then so does D.

Diagrammatic Pseudomorphism Coherence



Slogan. When T admits *UPC* such that Δ is an equivalence, then commutativity of *formal diagrams* for f reduces to commutativity of the *dissolution diagrams* in TG'. (use algebra coherence)

Lifts of plain/braided/symmetric structure morphisms:
Λ sends them to corresponding morphisms in X'
Δ sends them to *identities* in TG'

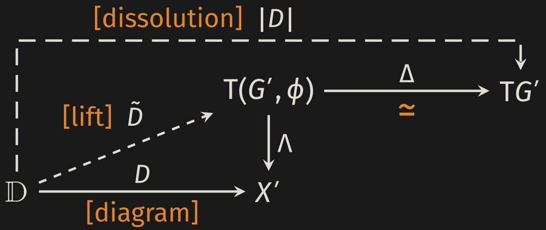
Pseudomorphism Coherence: Example 2

(from before)

$$
\begin{array}{ccc}
f(a)\cdot f(b)\cdot f(c)\cdot f(d) & \xrightarrow{f_2\cdot f_2} & f(a+b)\cdot f(c+d) \\
{\scriptstyle 1\cdot\beta\cdot 1}\downarrow & & \downarrow{\scriptstyle f_2} \\
f(a)\cdot f(c)\cdot f(b)\cdot f(d) & D & f(a+b+c+d) \\
{\scriptstyle f_2\cdot f_2}\downarrow & & \downarrow{\scriptstyle f(1+\beta+1)} \\
f(a+c)\cdot f(b+d) & \xrightarrow{f_2} & f(a+c+b+d)
\end{array}
$$

$$
\begin{array}{ccc}
\big(f(a),\ f(b),\ f(c),\ f(d)\big) & \xrightarrow{1} & \big(f(a),\ f(b),\ f(c),\ f(d)\big) \\
{\scriptstyle (1,\beta,1)}\downarrow & & \downarrow{\scriptstyle 1} \\
\big(f(a),\ f(c),\ f(b),\ f(d)\big) & |D| & \big(f(a),\ f(b),\ f(c),\ f(d)\big) \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle (1,\beta,1)} \\
\big(f(a),\ f(c),\ f(b),\ f(d)\big) & \xrightarrow{1} & \big(f(a),\ f(c),\ f(b),\ f(d)\big)
\end{array}
$$

Pseudomorphism Coherence



Interpretation: In each formal diagram *D*, one can apply naturality and other axioms to separate into two parts:

- one part commutes by axioms for *f*
- other part depends on axioms for T-algebras

Δ filters out first part, reduces to second part

Slogan (again). When Δ is an equivalence, commutativity of a formal diagram for *f* reduces to commutativity of the *dissolution diagram* in a free algebra.

Pseudomorphism Coherence: Example 3

Consider $f \cdot f : A \to A'$; $\quad (f \cdot f)(a) = f(a) \cdot f(a)$.
(f braided $\Rightarrow$ $f \cdot f$ plain monoidal)

Formal diagram for f:

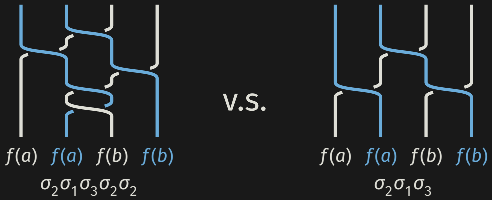
$$\begin{array}{ccc}
(f \cdot f)(a) \cdot (f \cdot f)(b) & & (f \cdot f)(a) \cdot (f \cdot f)(b) \\
f(a) \cdot f(a) \cdot f(b) \cdot f(b) & \xrightarrow{\beta \cdot \beta} & f(a) \cdot f(a) \cdot f(b) \cdot f(b) \\
{\scriptstyle 1 \cdot \beta \cdot 1} \downarrow & & \downarrow {\scriptstyle 1 \cdot \beta \cdot 1} \\
f(a) \cdot f(b) \cdot f(a) \cdot f(b) & & f(a) \cdot f(b) \cdot f(a) \cdot f(b) \\
{\scriptstyle f_2 \cdot f_2} \downarrow & & \downarrow {\scriptstyle f_2 \cdot f_2} \\
f(a+b) \cdot f(a+b) & \xrightarrow{\beta} & f(a+b) \cdot f(a+b) \\
(f \cdot f)(a+b) & & (f \cdot f)(a+b)
\end{array}$$

(monoidal naturality for $\beta : f \cdot f \to f \cdot f$)

Pseudomorphism Coherence: Example 3

Dissolution:

$$\begin{array}{ccc}
\big(f(a)\,,\ f(a)\,,\ f(b)\,,\ f(b)\big) & \xrightarrow[\sigma_1\sigma_3]{(\beta\,,\ \beta)} & \big(f(a)\,,\ f(a)\,,\ f(b)\,,\ f(b)\big) \\
{\scriptstyle (1\,,\ \beta\,,\ 1)}\big\downarrow{\scriptstyle \sigma_2} & & {\scriptstyle \sigma_2}\big\downarrow{\scriptstyle (1\,,\ \beta\,,\ 1)} \\
\big(f(a)\,,\ f(b)\,,\ f(a)\,,\ f(b)\big) & & \big(f(a)\,,\ f(b)\,,\ f(a)\,,\ f(b)\big) \\
{\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\
\big(f(a)\,,\ f(b)\,,\ f(a)\,,\ f(b)\big) & \xrightarrow[\beta]{\sigma_2\sigma_1\sigma_3\sigma_2} & \big(f(a)\,,\ f(b)\,,\ f(a)\,,\ f(b)\big)
\end{array}$$



distinct as braids; equal as permutations

Conclusion

Slogan (again). When Δ is an equivalence, commutativity of a formal diagram for f reduces to commutativity of the *dissolution diagram* in a free algebra.

That's what we do in:

Universal pseudomorphisms,
with applications to diagrammatic coherence
for braided and symmetric monoidal functors
joint with N. Gurski

https://arxiv.org/abs/2312.11261

Thank You!

References and Related Work

JS93 Joyal-Street (1993). *Braided tensor categories*. doi:10.1006/aima.1993.1055

ML98 Mac Lane (1998). *Categories for the working mathematician*. doi:10.1007/978-1-4757-4721-8

- Epstein (1966), Lewis (1974): Coherence for lax plain/symmetric monoidal functors (unit subtleties!)
- Blackwell-Kelly-Power (1989): Essential 2-monad theory; pseudomorphism classifiers
- Lack (2002): 2-monadic approach; coherence for pseudo*algebras*
- Malkiewich-Ponto (2022): general approach to diagrammatic coherence for algebras

Lax monoidal non-example [Lewis]

The left hand formal diagram for a lax monoidal functor
$f : (A, +, I) \to (A', \cdot, I')$
does not generally commute.

$$\begin{array}{ccc}
f(I) & \xrightarrow[\cong]{\lambda^{-1}} & I' \cdot f(I) \\
{\scriptstyle \rho^{-1}}\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle f_0 \cdot 1} \\
f(I) \cdot I' & \xrightarrow{1 \cdot f_0} & f(I) \cdot f(I)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow[\cong]{\lambda^{-1}} & 1 \times \mathbb{Z} \\
{\scriptstyle \rho^{-1}}\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle f_0 \cdot 1} \\
\mathbb{Z} \times 1 & \xrightarrow{1 \cdot f_0} & \mathbb{Z} \times \mathbb{Z}
\end{array}$$

Non-Example. The diagram at right does not commute
when $f = \mathrm{u} : (\mathcal{Ab}, \otimes, \mathbb{Z}) \to (\mathcal{Set}, \times, 1)$.