Graded Picard categories and the 2-type of the sphere

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joint work with N. Gurski and A.M. Osorno

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Abstract

The Stable Homotopy Hypothesis connects symmetric monoidal algebra and stable homotopy theory. In a classical special case, the tensor product of graded abelian groups leads to a model for the 1-dimensional stable homotopy of the sphere spectrum. In that case, the nontrivial sign rule for commuting elements corresponds to nontriviality in the first Postnikov invariant of the sphere.

This talk describes a generalization to graded Picard categories, modeling the 2-dimensional stable homotopy of the sphere. The possible sign rules for this case reduce to certain elementary permutations. We give a classification and describe the corresponding homotopy theory. The work we present is joint with N. Gurski and A.M. Osorno.

The Stable Homotopy Hypothesis

A guiding principle for development of higher-dimensional algebra.

Homotopy Hypothesis (Grothendieck). By taking *classifying spaces* and *fundamental n-groupoids*, there is an equivalence between the theory of weak n-groupoids and that of homotopy n-types.

(classifying space = geometric realization of nerve)

(status depends on definition of weak n-groupoid)

(algebraic definitions for small n (maybe ≤ 4?))

(∞-categorical versions hold by definition)

Key Data: Isomorphism classes in dimension $k \leftrightarrow$ homotopy group π_k.

The Stable Homotopy Hypothesis

Stable Homotopy Hypothesis. A refinement.

1. Symmetric monoidal structure corresponds to topological stability.
2. Braiding/symmetry data corresponds to Postnikov data.

   (cohomological connecting data)

(here too, status depends on definitions one uses on the categorical side)

For fully algebraic definitions:

- Dimension 0: Abelian groups $\leftrightarrow$ Eilenberg-Mac Lane spectra

  ($HA = \{K(A, n)\}$; sequence of Eilenberg-Mac Lane spaces)

  (one stable homotopy group)

- Dimension 1: Picard categories $\leftrightarrow$ stable 1-types.

  (known in folklore; appears in 2012 paper of J.-Osorno)

- Dimension 2: Theorem of Gurski-J.-Osorno (2019)

Today's goal

Explain

- Picard categories
- tensor product
- correspondence with low-dimensional stable homotopy (sketch)
- graded sign rule ↔ model for sphere in dimensions 0,1,2

Picard Groups and Picard Categories

Picard group of a commutative ring R: Group of invertible R-modules, up to isomorphism
I is *invertible*, with inverse J, if $I \otimes_R J \cong R$

(related to line bundles over Spec(R))

Examples.

- pic($\mathbb{Z}$) = {$\mathbb{Z}$} (the trivial Abelian group)
- pic($\mathbb{Z}[\sqrt{-5}]$) $\cong \mathbb{Z}/2$
- ...many more from commutative algebra and algebraic geometry

  (connections to algebraic K-theory)

Picard *category* of R: Category of invertible R-modules and isomorphisms, Pic(R)

- Tensor product makes Pic(R) symmetric monoidal.
- Monoidal unit is R.

Picard Groups and Picard Categories

Picard *category* of R: Category of invertible R-modules and isomorphisms, $\mathrm{Pic}(R)$

- Tensor product makes $\mathrm{Pic}(R)$ symmetric monoidal.
- Monoidal unit is R.
- $\mathrm{Pic}_0(R) = \mathrm{Pic}(R)/\text{isom.} = \mathrm{pic}(R)$
- $\mathrm{Pic}_1(R) = \mathrm{Aut}(R) = R^\times$
- Symmetry isomorphism (a.k.a. braiding) for each invertible I:

$$R \cong I \otimes J \xrightarrow[\cong]{\beta} J \otimes I \cong R$$

This composite, for each I, defines a function

$$\mathrm{Pic}_0(R) \longrightarrow \big(\,\mathrm{Pic}_1(R)\,\big)_{\text{2-torsion}}$$

(appears, e.g., in Galois theory of commutative rings)

Picard Categories: A Generalization

Definition

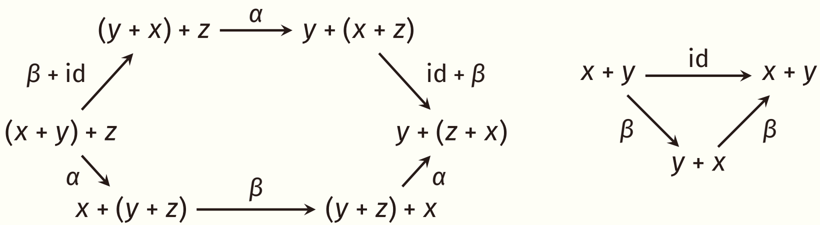
A *Picard category* is a symmetric monoidal category $(A, +, 0)$ in which

- each object x is invertible,
  ($x + \overline{x} \cong 0$, with triangle axioms)
- each morphism is an isomorphism.

Recall: A symmetric monoidal category $(A, +, 0, \beta)$ consists of:
a category A, a *monoidal sum* functor $+$, a *monoidal unit* 0, and a *braiding* natural isomorphism β, satisfying axioms for
associativity (pentagon), unity (left and right), and symmetry (hexagon and triangle).

Picard Categories: A Generalization

Symmetry axioms for braiding give a *symmetric cocycle condition*.



$$\beta(x,y) + \alpha(y,x,z) + \beta(x,z) = \alpha(y,z,x) + \beta(x,y+z) + \alpha(x,y,z)$$

$$\beta(x,y) + \beta(y,x) = 0$$ (additive identity)

Picard Categories: A Generalization

Theorem. (Sính; Joyal-Street) Picard categories are classified by symmetric 3-cocycles

$$(\alpha, \beta) \in H^3_{\mathrm{symm}}(P_0; P_1)$$

where P_0 = objects/isom. and P_1 = autom. of unit.

(classification is more general than stated here)

Every symmetric monoidal category M has an associated Picard category $\mathrm{Pic}(M)$ of invertible objects and morphisms.

For a commutative ring R,

$$\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Mod}_R).$$

Graded Abelian Groups

Consider the category of $\mathbb{Z}$-graded Abelian groups, $\mathtt{grAb}$. What is $\mathrm{Pic}(\mathtt{grAb})$?
Recall: $\mathrm{Pic}_0(\mathbb{Z}) = 0$; $\mathrm{Pic}_1(\mathbb{Z}) = \mathbb{Z}/2$.

If A and B are graded Abelian groups and $A \otimes B \cong \mathbb{Z}[0]$, then

$$A \cong \mathbb{Z}[n] \quad \text{and} \quad B \cong \mathbb{Z}[-n]$$

for some integer n.

(Notation: $\mathbb{Z}[n]$ denotes $\mathbb{Z}$ in degree n.)

The Picard Category of Graded Abelian Groups

Isomorphism classes of objects:

$$\pi_0\big(\mathrm{Pic}(\mathtt{grAb})\big) = \mathbb{Z}.$$

$$\mathbb{Z}[n] \leftrightarrow n$$

Automorphisms of unit:

$$\pi_1\big(\mathrm{Pic}(\mathtt{grAb})\big) = \mathbb{Z}^\times = \mathbb{Z}/2.$$

Symmetry:

$$\begin{array}{ccc} \mathbb{Z}[m] \otimes \mathbb{Z}[n] & \xrightarrow{\beta} & \mathbb{Z}[n] \otimes \mathbb{Z}[m] \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{Z}[m+n] & \xrightarrow{(-1)^{mn}} & \mathbb{Z}[n+m] \end{array}$$

The symmetry corresponds to the nontrivial morphism

$$\mathbb{Z} \to \mathbb{Z}/2.$$

Graded Abelian Groups

Define a Picard category $\mathbb{Z}^{\langle 1 \rangle}$:

Objects: Integers.

(Think: $n \leftrightarrow$ signed set with n elements)

Morphisms: $\hom(n, n) = \mathbb{Z}/2$;

$\hom(m, n) = \varnothing$ for $m \neq n$

(Think: sign of permutation)

Call $\mathbb{Z}^{\langle 1 \rangle}$ *super integers*. Observation: $\mathrm{Pic}(\mathtt{grAb}) \simeq \mathbb{Z}^{\langle 1 \rangle}$

The 2-Type of the Sphere

Freudenthal Suspension Theorem

For a CW-complex X, the sequence

$$\pi_n X \to \pi_{n+1} \Sigma X \to \pi_{n+2} \Sigma^2 X \to \cdots$$

eventually stabilizes.

Definition

The kth *stable homotopy group* of X is

$$\pi_n^s X = \lim_{k \to \infty} \pi_{n+k} \Sigma^k X.$$

The *suspension spectrum* of X is

$$\Sigma^\infty X = \{X, \Sigma X, \Sigma^2 X, \ldots\}.$$

The 2-Type of the Sphere

Example

The *sphere spectrum* is

$$\mathbf{S} = \Sigma^\infty S^0 = \{S^0, S^1, S^2, \dots\}.$$

Its homotopy groups are the stable homotopy groups

$$\pi_n \mathbf{S} = \lim_{k \to \infty} \pi_{n+k} S^k$$

The stable homotopy groups of the sphere spectrum have deep connections to algebra and geometry.

(For example, powerful invariants take values in $\pi_* \mathbf{S}$.)

The 2-Type of the Sphere

For low-dimensional computations, use the stable Postnikov tower.

(Either work in a category of spectra, or work with S^k for $k >> 0$.)

$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow & & \\ H(\pi_2, 2) & \xrightarrow{i_2} & P_2 & \xrightarrow{k_2} & H(\pi_3, 4) \\ & & \downarrow & & \\ H(\pi_1, 1) & \xrightarrow{i_1} & P_1 & \xrightarrow{k_1} & H(\pi_2, 3) \\ & & \downarrow & & \\ & & P_0 & \xrightarrow{k_0} & H(\pi_1, 2) \end{array}$$

(tower of fibrations)

composites $k_n i_n \in H^*(\pi_n; \pi_{n+1})$

$H(\pi, n)$: Eilenberg-Mac Lane spectrum with homotopy group π in dimension n

$P_n = P_n\mathbf{S}$: Postnikov truncation

$\pi_n = \pi_n\mathbf{S}$

$\pi_0 = \mathbb{Z}$ $\quad \pi_1 = \mathbb{Z}/2$

$\pi_2 = \mathbb{Z}/2$ $\quad \pi_3 = \mathbb{Z}/24 \ldots$

Tensor Product of Picard Categories

For Picard categories $A = (A, +, 0)$ and $B = (B, +, 0)$, define a Picard category $A \otimes B$.

Objects. Generated under + by formal pairs $a.b$, together with a single new object 0. ($a \in A$ and $b \in B$)

Morphisms. Generated by pairs, $f.g$, together with new natural isomorphisms:

$$a.b + a.d \xrightarrow{\delta_L} a.(b + d) \quad \text{and} \quad a.b + c.b \xrightarrow{\delta_R} (a + c).b$$

(distributivity)

$$a.b + c.d \xrightarrow{\beta} c.d + a.b$$

(braiding)

$$0 \xrightarrow{\zeta_R} a.0 \quad \text{and} \quad 0 \xrightarrow{\zeta_L} 0.b$$

(zero-ers a.k.a. nullifiers)

Tensor Product: Data satisfy five axioms

1. $a.(-)$ and $(-).b$ are strong monoidal
with monoidal constraints given by δ_L and δ_R

2. FOIL = FIOL

$$\begin{array}{ccc}
(a.b + c.b) + (a.d + c.d) & \xrightarrow{1 + \beta + 1} & (a.b + a.d) + (c.b + c.d) \\
\downarrow & & \downarrow \\
(a + c).b + (a + c).d & & a.(b + d) + c.(b + d) \\
& \searrow \quad (a + c).(b + d) \quad \swarrow &
\end{array}$$

3. nullifiers agree at zero

$$0 \underset{\zeta_L}{\overset{\zeta_R}{\rightrightarrows}} 0.0$$

Tensor Product: Data satisfy five axioms

4. braidings compatible with distributivity

$$\begin{array}{ccc} a.b + a.d & \xrightarrow{\beta} & a.d + a.b \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ a.(b + d) & \xrightarrow[a.\beta^B]{} & a.(d + b) \end{array} \qquad \begin{array}{ccc} a.b + c.b & \xrightarrow{\beta} & c.b + a.b \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ (a + c).b & \xrightarrow[\beta^A.b]{} & (c + a).b \end{array}$$

5. nullifiers compatible with distributivity

$$\begin{array}{ccc} 0 + 0 & \xrightarrow{=} & 0 \\ \downarrow{\scriptstyle \zeta + \zeta} & & \downarrow{\scriptstyle \zeta} \\ a.0 + c.0 & \xrightarrow[\delta]{} & (a + c).0 \end{array} \qquad \begin{array}{ccc} 0 + 0 & \xrightarrow{=} & 0 \\ \downarrow{\scriptstyle \zeta + \zeta} & & \downarrow{\scriptstyle \zeta} \\ 0.b + 0.d & \xrightarrow[\delta]{} & 0.(b + d) \end{array}$$

(this definition doesn't depend on invertibility; can be made more general)

Tensor Product Properties

Bilinearity. There are isomorphisms of categories

$$[A \otimes B, C] \cong \mathrm{Bilin}(A, B\,;\, C) \cong \langle A, \langle B, C\rangle\rangle$$

- $[A, B]$ denotes the category of *strict* monoidal functors,
- $\mathrm{Bilin}(A, B\,;\, C)$ denotes the category of functors that are *strong* monoidal in each variable, and
- $\langle A, B\rangle$ denotes the category of *strong* monoidal functors and transformations.

Commuting with coproducts. There are equivalences of categories

$$\left(\oplus_i A_i\right) \otimes B \simeq \oplus_i (A_i \otimes B).$$

(not isomorphisms of categories)

No New Invertibles. If $A \otimes B \simeq \mathbb{Z}^{\langle 1\rangle}$, then $A \simeq \mathbb{Z}^{\langle 1\rangle} \simeq B$.

(Picard *group* of Picard *categories* is trivial)

Tensor Product Properties

Relation to Abelian groups. If C and D are Abelian groups, regarded as Picard categories, then

$$\big(C \otimes D\big)_0 \cong C \otimes_{\mathbb{Z}} D \qquad \text{(tensor as } \mathbb{Z}\text{-modules)}$$

$$\big(C \otimes D\big)_1 \cong \mathrm{Tor}_1^{\mathbb{Z}}(C, D) \qquad \text{(degree 1 of derived tensor product)}$$

Symmetric Monoidal. The 2-category of Picard categories is symmetric monoidal as a *bicategory*.

(not symmetric monoidal in Cat-enriched sense.)

Inverses. Each generating object $a.b \in A \otimes B$ has two inverses: $\overline{a}.b$ and $a.\overline{b}$. There are *two canonical* isomorphisms

$$a.\overline{b} \xrightarrow{\cong} \overline{a}.b.$$

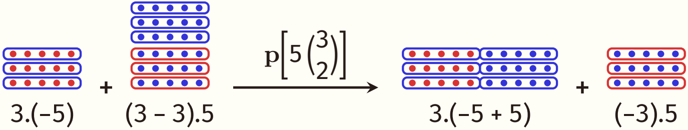
(induced by either $a + \overline{a} \cong 0$ or $b + \overline{b} \cong 0$)

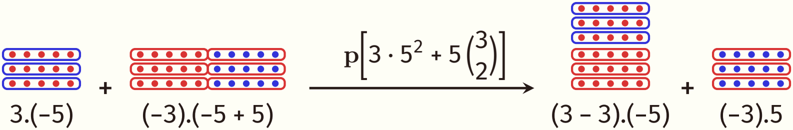
Example in $\mathbb{Z}^{\langle 1\rangle} \otimes \mathbb{Z}^{\langle 1\rangle} \simeq \mathbb{Z}^{\langle 1\rangle}$

Two canonical isomorphisms $3.(-5) \xrightarrow{\cong} (-3).5$

(generally distinct because left/right distributivity isomorphisms are distinct)

($\mathbf{p}[N] \in \mathbb{Z}/2$ denotes pairity of a number N, as automorphism in $\mathbb{Z}^{\langle 1\rangle}$.)





As permutations of -15 dots, these have different signs.

(summand $3 \cdot 5^2$ comes from distributing −3 over −5 + 5)

(apparent asymmetry is from our convention for interpreting rows v.s. columns)

Graded Picard Categories

We want to extend the tensor product to $\mathbb{Z}$-graded Picard categories.

- $(A \otimes B)_n = \bigoplus_{i+j=n} (A_i \otimes B_j)$
- Unit and associativity data given by extending ungraded case.
- Braiding ($\mathrm{M}(a) = \overline{a}$)

$$A_i \otimes B_j \xrightarrow{\tau} B_j \otimes A_i \xrightarrow{\mathrm{M}^{ij} \otimes \mathrm{id}} B_j \otimes A_i$$

- Structure isomorphisms $\beta^2 \cong \mathrm{id}$ (and others) require choice of isomorphisms

$$(\mathrm{M}^{ij} a).(\mathrm{M}^{ij} b) \xrightarrow{\cong} a.b$$

  and more.
- *Neither of the two canonical choices will work.*

  (no single choice will satisfy the necessary axioms)

  (some mix is needed)

Symmetric Monoidal Bicategories

Data. A symmetric monoidal bicategory B consists of

- A bicategory B with pseudofunctors
  $\oplus \colon B \times B \to B$ and $1 \colon * \to B$
- 1-cells for associator a and left/right unitors ℓ, r,
- 2-cells for pentagon and 3 unity diagrams: π, λ, μ, ρ,
  (π, λ, and ρ come from axioms of a monoidal 1-category)
- braiding 1-cells β with invertible 2-cells R for two hexagon isomorphisms,
  ($R_{x|y,z} \leftrightarrow$ passing two objects over one other)
  ($R_{x,y|z}$ is the opposite)
- a syllepsis isomorphism $v \colon \beta^2 \to \mathrm{id}$.

Symmetric Monoidal Bicategories

Axioms. The data $(B, \oplus, 1), (a, \ell, r), (\pi, \lambda, \mu, \rho)$ satisfy three axioms:

- left and right normalization for the 2-unitors λ, μ, ρ, and
- nonabelian 4-cocycle condition for π

The data $(\beta, R_{-|--}, R_{--|-}, v)$ satisfy seven axioms:

- four axioms for braided structure:

  three 4-strand axioms: •(• • •) (• • •) • (••)(••)

  Yang-Baxter (Reidemeister) for 3 strands
- three axioms for syllepsis

  related to 3-strand virtual crossings

Symmetric Monoidal Bicategories

Comments.

- Every bicategory is equivalent to a strict 2-category, but the corresponding statement for symmetric monoidal structures is false.

  (obstruction appears in corresponding classifying spaces)

  (e.g., the 2-category of Picard categories with its tensor product)

- Some data of symmetric monoidal bicategory comes from axioms in 1-dimensional case, but some is new.

  (e.g., the middle unitor, μ)

- For graded Picard categories, axioms reduce to certain equations in $\mathbb{Z}^{\langle 1 \rangle}$.

  (because objects are detected by morphisms out of $\mathbb{Z}^{\langle 1 \rangle}$)

  (reminiscent of some (symmetric) cocycle conditions)

Higher Sign Conventions

Consider up to four different Picard categories, in four different degrees: $A_{\mathtt{i}}$, $B_{\mathtt{j}}$, $C_{\mathtt{k}}$, $D_{\mathtt{l}}$.

Need two functions: $[\mathtt{i}|\mathtt{j},\mathtt{k}]$ and $\langle \mathtt{i}, \mathtt{j}\rangle$. (valued in $\mathbb{Z}/2$)

Data.

- $\beta_{\mathtt{i},\mathtt{j}} = (M^{\mathtt{ij}} \otimes \mathrm{id}) \circ \tau$
- $R_{\mathtt{i}|\mathtt{j},\mathtt{k}}$ = a combination of canonical isomorphisms having sign $[\mathtt{i}|\mathtt{j},\mathtt{k}] \in \mathbb{Z}/2$
- $R_{\mathtt{i},\mathtt{j}|\mathtt{k}} = \mathrm{id}$
- $v_{\mathtt{i},\mathtt{j}}$ = a combination of canonical isomorphisms having sign $\langle \mathtt{i}, \mathtt{j}\rangle \in \mathbb{Z}/2$

where $[\mathtt{i}|\mathtt{j},\mathtt{k}]$ and $\langle \mathtt{i}, \mathtt{j}\rangle$ satisfy the following equations in $\mathbb{Z}/2$.

Higher Sign Conventions

Braid Axioms.

1. $[i|j,k] + [i|j+k,l] = [i|k,l] + [i|j,k+l]$
2. (trivial when $R_{i,j|k} = \mathrm{id}$)
3. $[i+j|k,l] = [i|k,l] + [j|k,l]$
4. $[j|k,l] = [j|l,k]$

Syllepsis and Symmetry Axioms.

1. $[i|j,k] + \langle i, j+k\rangle = ikij + \langle i,k\rangle + \langle i,j\rangle$
2. $[j|i,k] + \langle i+j, k\rangle = ikjk + \langle i,k\rangle + \langle j,k\rangle$
3. $\langle i,j\rangle = \langle j,i\rangle$

Two Nontrivial Possibilities. (at least)

1. $\langle i,j\rangle = ij$ and $[i|j,k] = ijk$
2. $\langle i,j\rangle = \binom{ij}{2}$ and $[i|j,k] = 0$

Conclusion

Theorem (Gurski-J.-Osorno)

1. There is a choice of sign convention that makes $(\mathtt{grPic}, \otimes)$ symmetric monoidal as a bicategory.

   (Question: are different choices equivalent?)

2. With either nontrivial sign convention above,

$$B\big(\mathrm{Pic}(\mathtt{grPic})\big) \simeq P_2\mathbf{S}.$$

$\mathrm{Pic}(\mathtt{grPic})$ is the Picard 2-category of invertible graded Picard categories

$\mathrm{Pic}_0(\mathtt{grPic}) \cong \mathbb{Z}$	$\mathrm{Pic}_1(\mathtt{grPic}) \cong \mathbb{Z}/2$	$\mathrm{Pic}_2(\mathtt{grPic}) \cong \mathbb{Z}/2$
invertibles $\mathbb{Z}^{\langle 1 \rangle}[n]$	$\mathrm{Aut}_{\mathrm{Pic}}\big(\mathbb{Z}^{\langle 1 \rangle}\big)$	$\mathrm{Aut}_{\mathrm{Ab}}\big(\mathbb{Z}\big)$

Nontriv. braiding $\Rightarrow$ nontriv. Postnikov invariants

Thank You!