Low-degree cohomology for finite groups of Lie type

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Overview

Low-degree cohomology of finite algebraic groups

- $SL_n(\mathbb{F}_q)$, $SO_n(\mathbb{F}_q)$, $Sp_{2n}(\mathbb{F}_q)$, etc.
    - $q = p^r$
- Simple coefficient module $M = L(\lambda)$.
    - λ below a fundamental dominant weight
- Modular case: characteristic p.
    - $char(M) \not| \, |G(\mathbb{F}_q)| \Rightarrow H^*(G(\mathbb{F}_q), M) = 0$.
- Small primes
    - new techniques are necessary.

Combinatorial, topological, and scheme-theoretic techniques applied to problems in cohomology of finite groups, Hopf algebras, Lie algebras.

Motivation

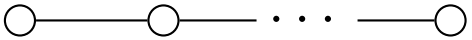
- Interest in finite group cohomology; modular case, small primes
- Generalize vanishing results of Cline-Parshall-Scott (1974)
  - Wiles's proof of Fermat's Last Theorem
- Reproduce and extend degree-two results
  - Avrunin (1978): certain minimal weights
  - Bell (1978): type A analyzed completely
- Relationship between finite and algebraic groups

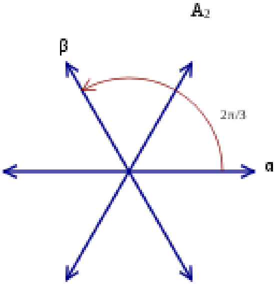
Algebraic Group Schemes

- k, algebraically closed field of positive characteristic p.
- G, (affine) algebraic group scheme over $k \leftrightarrow$ Hopf algebra $k[G]$.
  - A scheme is a geometric object, parametrizing (matrix) groups over k-algebras: $(SL_n(R), SO_n(R), Sp_{2n}(R))$.
- M, (rational) G-module $\leftrightarrow$ comodule over $k[G]$;
- Simple, simply-connected algebraic groups: classified by Lie type (Dynkin diagrams $\leftrightarrow$ root systems, Φ)
  - A_n, B_n, C_n, D_n; rank $n \geq 1$
  - G_2, F_4, E_6, E_7, E_8

Example: $A_n = SL_n$

$$SL_n(R) = \{(a_{ij}) | det(a_{ij}) = 1\}$$





$$B(R) = \begin{pmatrix} * & & \ast \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

$$U(R) = \begin{pmatrix} 1 & & \ast \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$T(R) = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

Wikipedia: `Root_system_A2.svg`

Group Cohomology

- Algebraic group cohomology:
  $H^*(G, M) = \mathrm{Ext}^*_G(k, M) = \mathrm{Ext}^*_{k[G]\text{-comod}}(k, M)$.
- Finite group cohomology:
  $H^*(G(\mathbb{F}_q), M) = \mathrm{Ext}^*_{G(\mathbb{F}_q)}(k, M) = \mathrm{Ext}^*_{kG(\mathbb{F}_q)}(k, M)$.
  - $M = M(\mathbb{F}_q)$
- Maximal torus $T \leq G$.
  - Simultaneous diagonalization of commuting matricies
    $\Rightarrow$ decomposition of representations into weight spaces
  - $X(T)$ = weight lattice;
    fundamental dominant weights $\omega_1, \ldots, \omega_n$
  - Weights are partially ordered.
- Highest-weight modules $M = L(\lambda)$, $\lambda \in X_+(T)$ (dominant weights).
  - Unique simple modules with highest-weight λ.

Example: $A_n = SL_n$

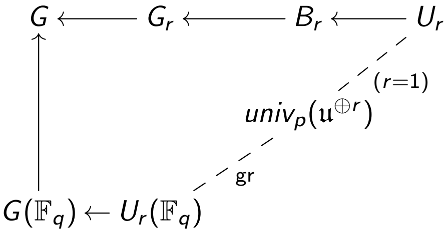
$$k[SL_n] = k[X_{ij}]/det - 1$$

Frobenius $F : (a_{ij}) \mapsto (a_{ij}^p)$

$$(SL_n)_r = ker\ F^r$$

If $R = \mathbb{F}_p$, $(SL_n(R))_1 = ker\ F = 1$;
more interesting when R has nilpotents, roots of unity.

Strategy: (top row)



Fact: simple module $L(\lambda)$ restricts to simple modules for $G(\mathbb{F}_q)$ and G_r.

Fundamental exact sequence

Consider the long exact sequence in cohomology induced by

$$0 \to k \to \mathcal{G}_r \to \mathcal{G}_r/k \to 0.$$

$$\begin{array}{llllll} 0 & \longrightarrow & \mathrm{Hom}_G(k, L(\lambda)) & \xrightarrow{\mathrm{res}_0} & \mathrm{Hom}_{G(\mathbb{F}_q)}(k, L(\lambda)) & \longrightarrow & \mathrm{Hom}_G(k, L(\lambda) \otimes \mathcal{G}_r/k) \\ & \longrightarrow & \mathrm{Ext}^1_G(k, L(\lambda)) & \xrightarrow{\mathrm{res}_1} & \mathrm{Ext}^1_{G(\mathbb{F}_q)}(k, L(\lambda)) & \longrightarrow & \mathrm{Ext}^1_G(k, L(\lambda) \otimes \mathcal{G}_r/k) \\ & \longrightarrow & \mathrm{Ext}^2_G(k, L(\lambda)) & \xrightarrow{\mathrm{res}_2} & \mathrm{Ext}^2_{G(\mathbb{F}_q)}(k, L(\lambda)) & \longrightarrow & \mathrm{Ext}^2_G(k, L(\lambda) \otimes \mathcal{G}_r/k) \\ & \longrightarrow & \cdots & & & & \end{array}$$

$$\mathcal{G}_r = \mathrm{ind}_{G(\mathbb{F}_q)}^G(k)$$

As a G-module, $\mathcal{G}_r/k$ admits a filtration with layers of the form $H^0(\mu) \otimes H^0(\mu^*)^{(r)}$.

Results: Comparison with algebraic group

Assume that

- $p > 2$ for $\Phi = A_n,\ D_n$.
- $p > 3$ for $\Phi = B_n,\ C_n,\ E_6,\ E_7,\ F_4$.
- $p > 5$ for $\Phi = E_8,\ G_2$.
- $q \geq 4$.

Theorem

Suppose $\lambda \leq \omega_j$ for some j. Then the restriction map

$$\mathrm{res}_t : H^t(G, L(\lambda)) \to H^t(G(\mathbb{F}_q), L(\lambda))$$

is an isomorphism for $t = 1$ and an injection for $t = 2$.

Results: Comparison with algebraic group

Assume the following Prime-Power Restrictions hold for p and $q = p^r$:

- $p > 3$ for $\Phi = A_n,\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ F_4$.
- $p > 5$ for $\Phi = E_8,\ G_2$.
- $q \geq 7$ for $\Phi = E_7,\ F_4$.

Theorem

Suppose $\lambda \leq \omega_j$ for some j, and suppose the Weight Condition holds for λ. Then the restriction map

$$\mathrm{res}_2 : H^2(G, L(\lambda)) \to H^2(G(\mathbb{F}_q), L(\lambda))$$

is an isomorphism.

We say that $\lambda \in X(T)_+$ satisfies the Weight Condition if

$$\max\left\{ -(\nu, \gamma^\vee) : \gamma \in \Delta,\ \nu \text{ a weight of } \operatorname{Ext}^1_{U_r}(k, L(\lambda)) \right\} < q.$$

Problem weights: Weight Condition fails to hold

Type	Weights
A_2, $q = 5$	ω_1, ω_2
B_n	$\alpha_0 = \omega_1$ (and $\widetilde{\alpha} = \omega_2$ if $n \geq 3$)
C_n	$\alpha_0 = \omega_2$
D_n	$\widetilde{\alpha} = \omega_2$
E_6	$\widetilde{\alpha} = \omega_2$
E_7	$\widetilde{\alpha} = \omega_1$
E_8	$\widetilde{\alpha} = \omega_8$
F_4	$\alpha_0 = \omega_4, \widetilde{\alpha} = \omega_1$
G_2	$\alpha_0 = \omega_1, \widetilde{\alpha} = \omega_2$

Table: Highest short roots are denoted by α_0, and highest long roots by $\widetilde{\alpha}$.

Finite group H^1, $\lambda = \omega_j$

Assume that

- $p > 2$ for $\Phi = A_n$, D_n
- $p > 3$ for $\Phi = B_n$, C_n, E_6, E_7, F_4, G_2
- $p > 5$ for $\Phi = E_8$

and assume $q \geq 4$.

Theorem

Then $H^1(G(\mathbb{F}_q), L(\omega_j)) = 0$ except for the following cases, in which we have $H^1(G(\mathbb{F}_q), L(\omega_j)) \cong k$:

- *Φ has type C_n, $n \geq 3$, $(n+1) = \sum_{i=0}^{t} b_i p^i$ with $0 \leq b_i < p$ and $b_t \neq 0$, and $j = 2b_i p^i$ for some $0 \leq i < t$ with $b_i \neq 0$;*
- *Φ is of type E_7, $p = 7$ and $j = 6$.*

Finite group H^1, $\lambda < \omega_j$ (exceptional types)

Let Φ be of exceptional type. Assume that

- $p > 3$ for $\Phi = E_6,\ F_4,\ G_2$.
- $p > 7$ for $\Phi = E_7, E_8$.

Theorem

Suppose $\lambda \leq \omega_j$ for some j. Then $H^1(G(\mathbb{F}_q), L(\lambda)) = 0$ except for the following cases, in which we have $H^1(G(\mathbb{F}_q), L(\lambda)) \cong k$:

- $\Phi = F_4$, $p = 13$, *and* $\lambda = 2\omega_4$.
- $\Phi = E_7$, $p = 19$, *and* $\lambda = 2\omega_1$.
- $\Phi = E_8$, $p = 31$, *and* $\lambda = 2\omega_8$.

Finite group H^2, $\lambda \leq \omega_j$

Assume that

- The Prime-Power Restrictions hold for p and q.
- $p > n$ for $\Phi = C_n$ if $\lambda = \omega_j$ with j even.
- For $\Phi = E_8$ and $p = 31$, $\lambda \neq \omega_7 + \omega_8$. ($H^2 \cong k$ in this case.)
- The Weight Condition holds for λ.

Theorem

Under the assumptions above, $H^2(G(\mathbb{F}_q), L(\lambda)) = 0$ except possibly the following cases:

- $\Phi = E_7$, $p = 5$, $\lambda = 2\omega_7$
- $\Phi = E_7$, $p = 7$, $\lambda = \omega_2 + \omega_7$
- $\Phi = E_8$, $p = 7$, $\lambda \in \{2\omega_7,\ \omega_1 + \omega_7,\ \omega_2 + \omega_8\}$
- $\Phi = E_8$, $p = 31$, $\lambda = \omega_6 + \omega_8$

Note: E_7 has 12 non-zero weights $\lambda \leq \omega_j$ for some j; E_8 has 23.

Finite Group H^2 for problem weights

Show, instead, that the restriction map vanishes.
The finite group cohomology is isomorphic to the term in column 3.

Theorem

Suppose that the Prime-Power Restrictions hold for p and q, and suppose that λ does not satisfy the Weight Condition. Assume moreover:

- *For $\Phi = B_n$ and $\lambda = \widetilde{\alpha}$, $\widetilde{\alpha}$ is not linked to α_0.*
- *For $\Phi = C_n$, $p \not| \ n$.*
- *For $p = q$ and $\Phi \neq A_2$, $p > 5$.*

Then

$$H^2(G(\mathbb{F}_q), L(\lambda) = \begin{cases} 0 & \text{if } \lambda = \alpha_0 \text{ and } \Phi \text{ has two root lengths} \\ k & \text{otherwise} \end{cases}$$

Summary of Methods: Fundamental exact sequence

$$\begin{array}{lllll} 0 & \longrightarrow & \mathrm{Hom}_G(k, L(\lambda)) & \xrightarrow{\mathrm{res}_0} & \mathrm{Hom}_{G(\mathbb{F}_q)}(k, L(\lambda)) & \longrightarrow & \mathrm{Hom}_G(k, L(\lambda) \otimes \mathcal{G}_r/k) \\ & \longrightarrow & \mathrm{Ext}^1_G(k, L(\lambda)) & \xrightarrow{\mathrm{res}_1} & \mathrm{Ext}^1_{G(\mathbb{F}_q)}(k, L(\lambda)) & \longrightarrow & \mathrm{Ext}^1_G(k, L(\lambda) \otimes \mathcal{G}_r/k) \\ & \longrightarrow & \mathrm{Ext}^2_G(k, L(\lambda)) & \xrightarrow{\mathrm{res}_2} & \mathrm{Ext}^2_{G(\mathbb{F}_q)}(k, L(\lambda)) & \longrightarrow & \mathrm{Ext}^2_G(k, L(\lambda) \otimes \mathcal{G}_r/k) \\ & \longrightarrow & \cdots & & & & \end{array}$$

$$\mathcal{G}_r = \mathrm{ind}_{G(\mathbb{F}_q)}^G(k)$$

As a G-module, $\mathcal{G}_r/k$ admits a filtration with layers of the form $H^0(\mu) \otimes H^0(\mu^*)^{(r)}$.

$$\mathrm{Ext}^i(k, L(\lambda) \otimes H^0(\mu) \otimes H^0(\mu^*)^{(r)}) \cong \mathrm{Ext}^i(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu)).$$

Two spectral sequences

Pass to Frobenius kernel G_r:

$$E_2^{i,j} = \mathrm{Ext}^i_{G/G_r}(V(\mu)^{(r)}, \mathrm{Ext}^j_{G_r}(k, M \otimes H^0(\mu))) \Rightarrow \mathrm{Ext}^{i+j}_G(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu))$$

Interchange induction and invariants:

$$E_2^{i,j} = R^i\mathrm{ind}_{B/B_r}^{G/G_r} \mathrm{Ext}^j_{B_r}(k, L(\lambda) \otimes \mu) \Rightarrow \mathrm{Ext}^{i+j}_{G_r}(k, L(\lambda) \otimes H^0(\mu))$$

Weight Condition implies low-degree vanishing of E_2, except in a handful of cases.

All types, all primes

Main Ideas

- Ascend from finite group to algebraic group
  - Analyze layers of column 3 with two spectral sequences
- In low degrees, G-cohomology is controlled by U_r-cohomology
  - U_r = Frobenius kernel, $ker(F^r)$ on unipotent $U \leq G$.
- Take torus invariants: T acts on $H^*(U_r, M)$
- Analyze socle layers of (torus-invariant) U_r-cohomology by weight
  semisimplicity
  vanishing of socle

Motivation:

- $U(\mathbb{F}_q)$ is the Sylow p-subgroup of $G(\mathbb{F}_q)$.
- $H^*(G, M)$ is T-invariant in $H^*(U_r, M)$.
- Previous work of this group on weights of U_r-cohomology.

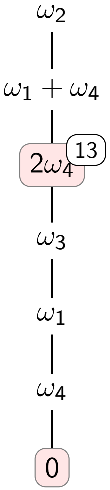
Cohomology of algebraic group

$H^*(G, L(\lambda)) = 0$ if either:

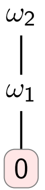
- $V(\lambda) \xrightarrow{\cong} L(\lambda)$ (types $A_n, B_n, D_n,\ \ p > 2$).
  - Weyl module $V(\lambda) \twoheadrightarrow L(\lambda)$.
- λ is not linked to 0 under the action of the affine Weyl group (Linkage Principle).
  - $0 \uparrow \lambda$ means $\lambda = w.0 + p\sigma$ for some $\sigma \in \mathbb{Z}\Phi$, $w \in W_p$.
  - Check $(1/p)\,(\lambda - w.0)$ for all $w \in W_p$.

Linkage results for exceptional types

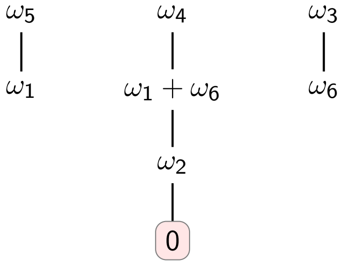
F_4



G_2

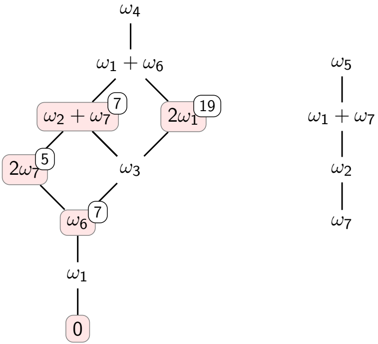


E_6

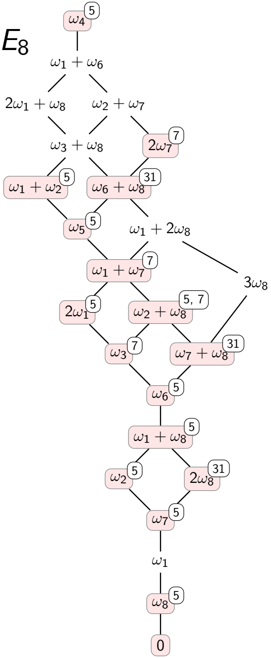


$5 \leq p$

E_7


ω4
ω1 + ω6
ω2 + ω7 7
2ω1 19
2ω7 5
ω3
ω6 7
ω1
0
ω5
ω1 + ω7
ω2
ω7

E_8


ω4 5
ω1 + ω6
2ω1 + ω8
ω2 + ω7
ω3 + ω8
2ω7 7
ω1 + ω2 5
ω6 + ω8 31
ω5 5
ω1 + 2ω8
ω1 + ω7 7
3ω8
2ω1 5
ω2 + ω8 5, 7
ω3 7
ω7 + ω8 31
ω6 5
ω1 + ω8 5
ω2 5
2ω8 31
ω7 5
ω1
ω8 5
0

$5 \leq p$

Type C: Kleshchev-Sheth

Work of Kleshchev-Sheth:

On extensions of simple modules over symmetric and algebraic groups & Corrigendum (1999 & 2001)

Complete description of $V(\omega_j) \twoheadrightarrow L(\omega_j)$ for type $C_n = Sp_{2n}$
Combinatorics depending on the base-p digits of $n+1$ and $n-j+1$.

- Nontrivial combinatorics; comparable to Young diagrams for symmetric groups.
- Number of simple composition factors of $V(\omega_j)$ may be exponential in j.
- Software to draw diagrams of $V(\omega_j)$, with information about $H^1(C_n, L(\omega_i))$ and $[V(\omega_i) : k]$.

Type C: Kleshchev-Sheth

Type C_n: there is a bijection (of posets) between the composition factors of the Weyl module $V(\omega_j)$ and $\widehat{A}_j$.

$$n - j + 1 = \sum_{i=0}^{t} c_i p^i, \quad 0 \leq c_i < p$$

Consider half-open $\mathbb{Z}$-intervals of the form $I = [a, b)$ with $c_a \neq 0$ and $c_b \neq p - 1$, and define

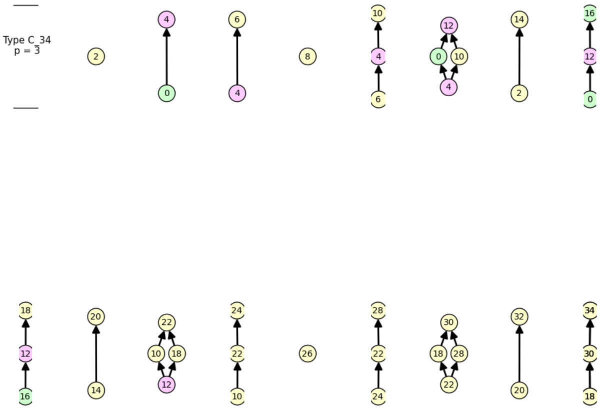
$$\delta(I) = p^a + \sum_{i=a}^{b-1} (p - 1 - c_i) p^i; \quad \delta(\emptyset) = 0$$

δ is additive on disjoint unions of intervals

$$\widehat{A}_j = \{I = [a_1, b_1) \cup \cdots \cup [a_t, b_t) \text{ such that } b_i < a_{i+1} \text{ and } 2\delta(I) \leq j\}$$

$$I \leftrightarrow L(\omega_i), \quad i = j - 2\delta(I)$$

Type C: Kleshchev-Sheth ($n = 34 = 1 + 6 + 0 + 27$)



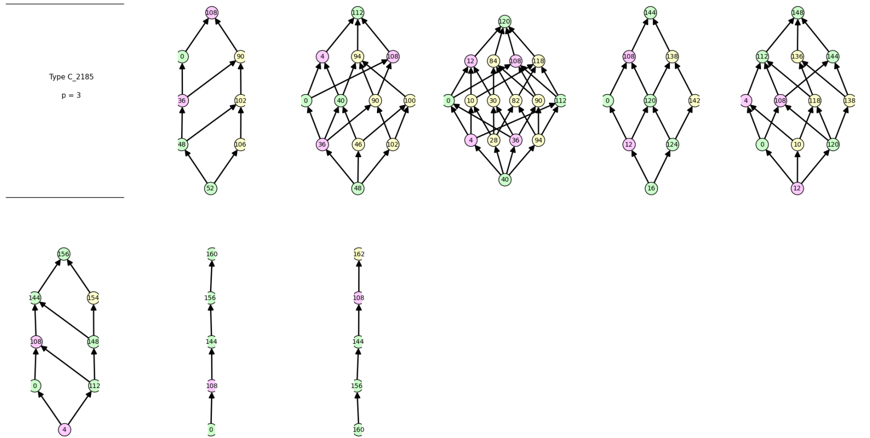
i $\longleftrightarrow$ $L(\omega_i)$

i $\mathrm{Ext}^1_{C_n}(k, L(\omega_i)) \cong k.$

i $[V(\omega_i) : k] = 1$

i neither

Type C: Kleshchev-Sheth ($n = 2185 = 3^7 - 2$)



Type C: Kleshchev-Sheth

All values of n and j for which $H^2(Sp_{2n}(\mathbb{F}_q), L(\omega_j)) \ncong 0$:
$p = 3, n < 40$

In each case, H^2 is 1-dimensional.

n	j
6	6
7	6
8	none
9	6
10	6
11	none
12	6
13	6
14	none

n	j
15	6, 8
16	6, 10
17	none
18	6, 14
19	6, 16
20	18
21	6, 18
22	6, 18
23	18

n	j
24	6, 8, 18
25	6, 10, 18
26	none
27	6, 14
28	6, 16
29	18
30	6, 18
31	6, 18
32	18

n	j
33	6, 8, 18
34	6, 10, 18
35	none
36	6, 14
37	6, 16
38	18

Type C: Kleshchev-Sheth

All values of n and j for which $H^2(Sp_{2n}(\mathbb{F}_q), L(\omega_j)) \ncong 0$:
$p = 5, n < 55$

In each case, H^2 is 1-dimensional.

n	j
10	10
11	10
12	10
13	10
14	none
15	10
16	10
17	10
18	10
19	none

n	j
20	10
21	10
22	10
23	10
24	none
25	10
26	10
27	10
28	10
29	none

n	j
30	10
31	10
32	10
33	10
34	none
35	10, 12
36	10, 14
37	10, 16
38	10, 18
39	none

n	j
40	10, 22
41	10, 24
42	10, 26
43	10, 28
44	none
45	10, 32
46	10, 34
47	10, 36
48	10, 38
49	none

n	j
50	10, 42
51	10, 44
52	10, 46
53	10, 48
54	50

Higher Cohomology: Lots of room to grow

For higher cohomology groups, there are many opportunities to become nontrivial.

- For large λ, $H^*(G, L(\lambda))$ is non-zero.
- Layers of $\mathcal{G}_r/k$ may have non-trivial higher cohomology.
- Passage from U_r to G_r to G may not induce isomorphisms.

Conclusion

Vanishing results for $H^t(G(\mathbb{F}_q), L(\lambda))$, $t = 1, 2$ with λ small and mild (single-digit) conditions on p except:

- $\lambda = \omega_j$ for type C_n and various even j.
  - H^1 completely understood.
  - H^2 only partially understood.
- Handful of special cases for exceptional types and for ~~problem~~ *special* weights.
  - Nonvanishing H^1 for $E_7, p = 7, \lambda = \omega_6$.
  - Nonvanishing H^1 for $p = h + 1$, $\lambda = 2\omega_j$ linked to 0 (3 cases).
  - Nonvanishing H^2 for, e.g., $\lambda = \widetilde{\alpha}$, $\Phi = D_n, E_n, F_4, G_2$ or $\lambda = \omega_1, \omega_2$, $\Phi = A_2$.

Combinatorial, topological, and scheme-theoretic techniques applied to problems in cohomology of finite groups, Hopf algebras, Lie algebras.

Thank You!