Low-degree cohomology for finite groups of Lie type

Niles Johnson
Joint with UGA VIGRE Algebra Group

Department of Mathematics
University of Georgia

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UGA VIGRE Algebra Group

**Faculty**
- Brian D. Boe
- Jon F. Carlson
- Leonard Chastkofsky
- Daniel K. Nakano
- Lisa Townsley

**Postdoctoral Fellows**
- Christopher M. Drupieski
- Niles Johnson
- Benjamin F. Jones

**Graduate Students**
- Brian Bonsignore
- Theresa Brons
- Adrian M. Brunyate
- Wenjing Li
- Phong Tanh Luu
- Tiago Macedo
- Nham Vo Ngo
- Duc Duy Nguyen
- Brandon L. Samples
- Andrew J. Talian
- Benjamin J. Wyser

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Overview

Low-degree cohomology of finite algebraic groups

- $SL_n(\mathbb{F}_q), SO_n(\mathbb{F}_q), Sp_{2n}(\mathbb{F}_q)$, etc.
  - $q = p^r$

- Simple coefficient module $M = L(\lambda)$.
  - $\lambda$ below a fundamental dominant weight

- Modular case: characteristic $p$.
  - $\text{char}(M) \not| |G(\mathbb{F}_q)| \Rightarrow H^*(G(\mathbb{F}_q), M) = 0$.

- Small primes
  - new techniques are necessary.

Combinatorial, topological, and scheme-theoretic techniques applied to problems in cohomology of finite groups, Hopf algebras, Lie algebras.
Interest in finite group cohomology; modular case, small primes

Generalize vanishing results of Cline-Parshall-Scott (1974)
  ▶ Wiles’s proof of Fermat’s Last Theorem

Reproduce and extend degree-two results
  ▶ Avrunin (1978): certain minimal weights
  ▶ Bell (1978): type A analyzed completely

Relationship between finite and algebraic groups
Algebraic Group Schemes

- $k$, algebraically closed field of positive characteristic $p$.

- $G$, (affine) algebraic group scheme over $k \leftrightarrow$ Hopf algebra $k[G]$.
  - A scheme is a geometric object, parametrizing (matrix) groups over $k$-algebras: $(SL_n(R), SO_n(R), Sp_{2n}(R))$.

- $M$, (rational) $G$-module $\leftrightarrow$ comodule over $k[G]$;

- Simple, simply-connected algebraic groups: classified by Lie type (Dynkin diagrams $\leftrightarrow$ root systems, $\Phi$)
  - $A_n, B_n, C_n, D_n$; rank $n \geq 1$
  - $G_2, F_4, E_6, E_7, E_8$
Example: $A_n = SL_n$

$SL_n(R) = \{(a_{ij}) | \det(a_{ij}) = 1\}$

$B(R) = \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix}$

$U(R) = \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix}$

$T(R) = \begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix}$

Wikipedia: Root_system_A2.svg
Group Cohomology

- Algebraic group cohomology:
  \[ H^* (G, M) = \text{Ext}^*_G (k, M) = \text{Ext}^*_k[G] \text{-comod} (k, M). \]

- Finite group cohomology:
  \[ H^* (G(\mathbb{F}_q), M) = \text{Ext}^*_G(\mathbb{F}_q)(k, M) = \text{Ext}^*_k G(\mathbb{F}_q)(k, M). \]
  ▶ \( M = M(\mathbb{F}_q) \)

- Maximal torus \( T \leq G \).
  ▶ Simultaneous diagonalization of commuting matrices
    \( \Rightarrow \) decomposition of representations into weight spaces
  ▶ \( X(T) = \) weight lattice;
    fundamental dominant weights \( \omega_1, \ldots, \omega_n \)
  ▶ Weights are partially ordered.

- Highest-weight modules \( M = L(\lambda), \lambda \in X_+(T) \) (dominant weights).
  ▶ Unique simple modules with highest-weight \( \lambda \).
Example: $A_n = SL_n$

$$k[SL_n] = k[X_{ij}]/\text{det} - 1$$

Frobenius $F : (a_{ij}) \mapsto (a_{ij}^p)$

$$(SL_n)_r = \ker F^r$$

If $R = \mathbb{F}_p$, $(SL_n(R))_1 = \ker F = 1$;
more interesting when $R$ has nilpotents, roots of unity.

Strategy: (top row)

\[ G \leftarrow G_r \leftarrow B_r \leftarrow U_r \]

Fact: simple module
$L(\lambda)$ restricts to simple modules for $G(\mathbb{F}_q)$ and $G_r$. 

\[ G(\mathbb{F}_q) \leftarrow U_r(\mathbb{F}_q) \]
Consider the long exact sequence in cohomology induced by

$$0 \to k \to G_r \to G_r/k \to 0.$$  

$$
\begin{array}{cccc}
0 & \to & \text{Hom}_G(k, L(\lambda)) & \xrightarrow{\text{res}_0} & \text{Hom}_G(\mathbb{F}_q, k, L(\lambda)) & \to & \text{Hom}_G(k, L(\lambda) \otimes G_r/k) \\
& \to & \text{Ext}^1_G(k, L(\lambda)) & \xrightarrow{\text{res}_1} & \text{Ext}^1_G(\mathbb{F}_q, k, L(\lambda)) & \to & \text{Ext}^1_G(k, L(\lambda) \otimes G_r/k) \\
& \to & \text{Ext}^2_G(k, L(\lambda)) & \xrightarrow{\text{res}_2} & \text{Ext}^2_G(\mathbb{F}_q, k, L(\lambda)) & \to & \text{Ext}^2_G(k, L(\lambda) \otimes G_r/k) \\
& \to & \ldots & & & & \\
\end{array}
$$

$$G_r = \text{ind}^G_G(\mathbb{F}_q)(k)$$

As a $G$-module, $G_r/k$ admits a filtration with layers of the form $H^0(\mu) \otimes H^0(\mu^*)(r)$. 
Assume that

- $p > 2$ for $\Phi = A_n, D_n$.
- $p > 3$ for $\Phi = B_n, C_n, E_6, E_7, F_4$.
- $p > 5$ for $\Phi = E_8, G_2$.
- $q \geq 4$.

**Theorem**

Suppose $\lambda \leq \omega_j$ for some $j$. Then the restriction map

$$\text{res}_t : H^t(G, L(\lambda)) \rightarrow H^t(G(\mathbb{F}_q), L(\lambda))$$

is an isomorphism for $t = 1$ and an injection for $t = 2$. 
Results: Comparison with algebraic group

Assume the following Prime-Power Restrictions hold for $p$ and $q = p^r$:

- $p > 3$ for $\Phi = A_n, B_n, C_n, D_n, E_6, E_7, F_4$.
- $p > 5$ for $\Phi = E_8, G_2$.
- $q \geq 7$ for $\Phi = E_7, F_4$.

**Theorem**

Suppose $\lambda \leq \omega_j$ for some $j$, and suppose the Weight Condition holds for $\lambda$. Then the restriction map

$$\text{res}_2 : H^2(G, L(\lambda)) \rightarrow H^2(G(\mathbb{F}_q), L(\lambda))$$

is an isomorphism.

We say that $\lambda \in X(T)_+$ satisfies the Weight Condition if

$$\max \left\{ -\langle \nu, \gamma^\vee \rangle : \gamma \in \Delta, \nu \text{ a weight of } \text{Ext}^1_{U_r}(k, L(\lambda)) \right\} < q.$$
### Problem weights: Weight Condition fails to hold

<table>
<thead>
<tr>
<th>Type</th>
<th>Weights</th>
</tr>
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<tbody>
<tr>
<td>$A_2$, $q = 5$</td>
<td>$\omega_1, \omega_2$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\alpha_0 = \omega_1$ (and $\tilde{\alpha} = \omega_2$ if $n \geq 3$)</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\alpha_0 = \omega_2$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\tilde{\alpha} = \omega_2$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\tilde{\alpha} = \omega_2$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\tilde{\alpha} = \omega_1$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$\tilde{\alpha} = \omega_8$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$\alpha_0 = \omega_4, \tilde{\alpha} = \omega_1$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\alpha_0 = \omega_1, \tilde{\alpha} = \omega_2$</td>
</tr>
</tbody>
</table>

**Table**: Highest short roots are denoted by $\alpha_0$, and highest long roots by $\tilde{\alpha}$. 
Finite group $H^1, \lambda = \omega_j$

Assume that

1. $p > 2$ for $\Phi = A_n, D_n$
2. $p > 3$ for $\Phi = B_n, C_n, E_6, E_7, F_4, G_2$
3. $p > 5$ for $\Phi = E_8$

and assume $q \geq 4$.

**Theorem**

Then $H^1(G(\mathbb{F}_q), L(\omega_j)) = 0$ except for the following cases, in which we have $H^1(G(\mathbb{F}_q), L(\omega_j)) \cong k$:

1. $\Phi$ has type $C_n$, $n \geq 3$, $(n + 1) = \sum_{i=0}^{t} b_ip^i$ with $0 \leq b_i < p$ and $b_t \neq 0$, and $j = 2b_ip^i$ for some $0 \leq i < t$ with $b_i \neq 0$;
2. $\Phi$ is of type $E_7$, $p = 7$ and $j = 6$. 
Finite group $H^1$, $\lambda < \omega_j$ (exceptional types)

Let $\Phi$ be of exceptional type. Assume that
- $p > 3$ for $\Phi = E_6, F_4, G_2$.
- $p > 7$ for $\Phi = E_7, E_8$.

**Theorem**

Suppose $\lambda \leq \omega_j$ for some $j$. Then $H^1(G(F_q), L(\lambda)) = 0$ except for the following cases, in which we have $H^1(G(F_q), L(\lambda)) \cong k$:
- $\Phi = F_4$, $p = 13$, and $\lambda = 2\omega_4$.
- $\Phi = E_7$, $p = 19$, and $\lambda = 2\omega_1$.
- $\Phi = E_8$, $p = 31$, and $\lambda = 2\omega_8$. 
Finite group $H^2, \lambda \leq \omega_j$

Assume that

- The **Prime-Power Restrictions** hold for $p$ and $q$.
- $p > n$ for $\Phi = C_n$ if $\lambda = \omega_j$ with $j$ even.
- For $\Phi = E_8$ and $p = 31$, $\lambda \neq \omega_7 + \omega_8$. ($H^2 \cong k$ in this case.)
- The **Weight Condition** holds for $\lambda$.

**Theorem**

*Under the assumptions above, $H^2(G(\mathbb{F}_q), L(\lambda)) = 0$ except possibly the following cases:*  

- $\Phi = E_7$, $p = 5$, $\lambda = 2\omega_7$
- $\Phi = E_7$, $p = 7$, $\lambda = \omega_2 + \omega_7$
- $\Phi = E_8$, $p = 7$, $\lambda \in \{2\omega_7, \omega_1 + \omega_7, \omega_2 + \omega_8\}$
- $\Phi = E_8$, $p = 31$, $\lambda = \omega_6 + \omega_8$

**Note:** $E_7$ has 12 non-zero weights $\lambda \leq \omega_j$ for some $j$; $E_8$ has 23.
Finite Group $H^2$ for problem weights

Show, instead, that the restriction map vanishes. The finite group cohomology is isomorphic to the term in column 3.

**Theorem**

Suppose that the **Prime-Power Restrictions** hold for $p$ and $q$, and suppose that $\lambda$ does not satisfy the **Weight Condition**. Assume moreover:

- For $\Phi = B_n$ and $\lambda = \tilde{\alpha}$, $\tilde{\alpha}$ is not linked to $\alpha_0$.
- For $\Phi = C_n$, $p \not| n$.
- For $p = q$ and $\Phi \neq A_2$, $p > 5$.

Then

$$H^2(G(\mathbb{F}_q), L(\lambda)) = \begin{cases} 0 & \text{if } \lambda = \alpha_0 \text{ and } \Phi \text{ has two root lengths} \\ k & \text{otherwise} \end{cases}$$
Summary of Methods: Fundamental exact sequence

\[
\begin{array}{cccc}
0 & \rightarrow & \text{Hom}_G(k, L(\lambda)) & \overset{\text{res}_0}{\rightarrow} & \text{Hom}_{G(\mathbb{F}_q)}(k, L(\lambda)) & \rightarrow & \text{Hom}_G(k, L(\lambda) \otimes \mathcal{G}_r/k) \\
& \rightarrow & \text{Ext}_G^1(k, L(\lambda)) & \overset{\text{res}_1}{\rightarrow} & \text{Ext}_G^1_{\mathbb{F}_q}(k, L(\lambda)) & \rightarrow & \text{Ext}_G^1(k, L(\lambda) \otimes \mathcal{G}_r/k) \\
& \rightarrow & \text{Ext}_G^2(k, L(\lambda)) & \overset{\text{res}_2}{\rightarrow} & \text{Ext}_G^2_{\mathbb{F}_q}(k, L(\lambda)) & \rightarrow & \text{Ext}_G^2(k, L(\lambda) \otimes \mathcal{G}_r/k) \\
& \rightarrow & \ldots
\end{array}
\]

\[
\mathcal{G}_r = \text{ind}_G^{\mathbb{F}_q}(k)
\]

As a \(G\)-module, \(\mathcal{G}_r/k\) admits a filtration with layers of the form \(H^0(\mu) \otimes H^0(\mu^*)(r)\).

\[
\text{Ext}^i(k, L(\lambda) \otimes H^0(\mu) \otimes H^0(\mu^*)(r)) \cong \text{Ext}^i(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu)).
\]
Two spectral sequences

Pass to Frobenius kernel $G_r$:

$$E_2^{i,j} = \text{Ext}^i_{G/G_r}(V(\mu)^{(r)}, \text{Ext}^j_{G_r}(k, M \otimes H^0(\mu)))$$
$$\Rightarrow \text{Ext}^{i+j}_{G}(V(\mu)^{(r)}, L(\lambda) \otimes H^0(\mu))$$

Interchange induction and invariants:

$$E_2^{i,j} = R^i \text{ind}_{B/B_r}^G \text{Ext}^j_{B_r}(k, L(\lambda) \otimes \mu)$$
$$\Rightarrow \text{Ext}^{i+j}_{G_r}(k, L(\lambda) \otimes H^0(\mu))$$

**Weight Condition** implies low-degree vanishing of $E_2$, except in a handful of cases.
All types, all primes

Main Ideas

- Ascend from finite group to algebraic group
  - Analyze layers of column 3 with two spectral sequences
- In low degrees, $G$-cohomology is controlled by $U_r$-cohomology
  - $U_r = \text{Frobenius kernel, } \ker(F^r)$ on unipotent $U \leq G$.
- Take torus invariants: $T$ acts on $H^*(U_r, M)$
- Analyze socle layers of (torus-invariant) $U_r$-cohomology by weight semisimplicity
  - vanishing of socle

Motivation:

- $U(\mathbb{F}_q)$ is the Sylow $p$-subgroup of $G(\mathbb{F}_q)$.
- $H^*(G, M)$ is $T$-invariant in $H^*(U_r, M)$.
- Previous work of this group on weights of $U_r$-cohomology.
Cohomology of algebraic group

\[ H^*(G, L(\lambda)) = 0 \text{ if either:} \]

- \( V(\lambda) \cong L(\lambda) \) (types \( A_n, B_n, D_n, \ p > 2 \)).
  - Weyl module \( V(\lambda) \rightarrow L(\lambda) \).

- \( \lambda \) is not linked to 0 under the action of the affine Weyl group (Linkage Principle).
  - \( 0 \uparrow \lambda \) means \( \lambda = w.0 + p\sigma \) for some \( \sigma \in \mathbb{Z}\Phi, w \in W_p \).
  - Check \( (1/p)(\lambda - w.0) \) for all \( w \in W_p \).
Linkage results for exceptional types

\[ F_4 \]

\[ G_2 \]

\[ E_6 \]

\[ 5 \leq p \]
Summary of Methods

Exceptional Type

\[ \omega_5 + \omega_1 + \omega_7 \]

\[ \omega_7 + \omega_2 \]

\[ 2\omega_1 + 19 \]

\[ \omega_2 + \omega_7 \]

\[ \omega_1 + \omega_6 \]

\[ \omega_4 \]

\[ 0 \]

\[ 5 \leq p \]

\[ E_8 \]

\[ \omega_4 + 5 \]

\[ \omega_1 + \omega_6 \]

\[ 2\omega_1 + \omega_8 \]

\[ \omega_2 + \omega_7 \]

\[ \omega_3 + \omega_8 \]

\[ 2\omega_7 + 7 \]

\[ \omega_1 + \omega_2 + 5 \]

\[ \omega_6 + \omega_8 + 31 \]

\[ \omega_5 + 5 \]

\[ \omega_1 + 2\omega_8 \]

\[ 2\omega_1 + 5 \]

\[ \omega_2 + \omega_8 + 5, 7 \]

\[ \omega_3 + 7 \]

\[ \omega_7 + \omega_8 + 31 \]

\[ \omega_6 + 5 \]

\[ \omega_1 + \omega_8 + 5 \]

\[ \omega_2 + 5 \]

\[ 2\omega_8 + 31 \]

\[ \omega_7 + 5 \]

\[ \omega_1 + 5 \]

\[ \omega_8 + 5 \]

\[ 0 \]

UGA VIGRE Algebra (UGA)  
Low-degree cohomology  
September 2011  
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Type C: Kleshchev-Sheth

Work of Kleshchev-Sheth:

*On extensions of simple modules over symmetric and algebraic groups & Corrigendum* (1999 & 2001)

Complete description of $V(\omega_j) \rightarrow L(\omega_j)$ for type $C_n = Sp_{2n}$

Combinatorics depending on the base-$p$ digits of $n + 1$ and $n - j + 1$.

- Nontrivial combinatorics; comparable to Young diagrams for symmetric groups.
- Number of simple composition factors of $V(\omega_j)$ may be exponential in $j$.
- Software to draw diagrams of $V(\omega_j)$, with information about $H^1(C_n, L(\omega_i))$ and $[V(\omega_i) : k]$. 

Type C: Kleshchev-Sheth

Type $C_n$: there is a bijection (of posets) between the composition factors of the Weyl module $V(\omega_j)$ and $\hat{A}_j$.

\[ n - j + 1 = \sum_{i=0}^{t} c_i p^i, \quad 0 \leq c_i < p \]

Consider half-open $\mathbb{Z}$-intervals of the form $I = [a, b)$ with $c_a \neq 0$ and $c_b \neq p - 1$, and define

\[ \delta(I) = p^a + \sum_{i=a}^{b-1} (p - 1 - c_i) p^i; \quad \delta(\emptyset) = 0 \]

$\delta$ is additive on disjoint unions of intervals

\[ \hat{A}_j = \{I = [a_1, b_1) \cup \cdots \cup [a_t, b_t) \text{ such that } b_i < a_{i+1} \text{ and } 2\delta(I) \leq j\} \]

\[ I \leftrightarrow L(\omega_i), \quad i = j - 2\delta(I) \]
Type C: Kleshchev-Sheth ($n = 34 = 1 + 6 + 0 + 27$)

$\text{Ext}_C^1(k, L(\omega_i)) \cong k.$

$[V(\omega_i) : k] = 1$

neither
Type $C$: Kleshchev-Sheth ($n = 2185 = 3^7 - 2$)
Type C: Kleshchev-Sheth

All values of $n$ and $j$ for which $H^2(Sp_{2n}(\mathbb{F}_q), L(\omega_j)) \neq 0$:

$p = 3, n < 40$

In each case, $H^2$ is 1-dimensional.

<table>
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<th>$j$</th>
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</tbody>
</table>
Type C: Kleshchev-Sheth

All values of \( n \) and \( j \) for which \( H^2(\text{Sp}_{2n}(\mathbb{F}_q), L(\omega_j)) \not\cong 0 \):

\( p = 5, \; n < 55 \)

In each case, \( H^2 \) is 1-dimensional.
For higher cohomology groups, there are many opportunities to become nontrivial.

- For large $\lambda$, $H^*(G, L(\lambda))$ is non-zero.
- Layers of $G_r/k$ may have non-trivial higher cohomology.
- Passage from $U_r$ to $G_r$ to $G$ may not induce isomorphisms.
Conclusion

Vanishing results for $H^t(G(\mathbb{F}_q), L(\lambda))$, $t = 1, 2$ with $\lambda$ small and mild (single-digit) conditions on $p$ except:

- $\lambda = \omega_j$ for type $C_n$ and various even $j$.
  - $H^1$ completely understood.
  - $H^2$ only partially understood.
- Handful of special cases for exceptional types and for problem special weights.
  - Nonvanishing $H^1$ for $E_7$, $p = 7$, $\lambda = \omega_6$.
  - Nonvanishing $H^1$ for $p = h + 1$, $\lambda = 2\omega_j$ linked to 0 (3 cases).
  - Nonvanishing $H^2$ for, e.g., $\lambda = \check{\alpha}$, $\Phi = D_n, E_n, F_4, G_2$ or $\lambda = \omega_1, \omega_2$, $\Phi = A_2$.

**Combinatorial, topological, and scheme-theoretic** techniques applied to problems in cohomology of **finite groups, Hopf algebras, Lie algebras**.

Thank You!