

# GALOIS NOTES

NILES JOHNSON

ABSTRACT. This is a collection of notes from our reading project on Galois theory for rings and ring spectra. An attempt is made to outline main ideas and give references, but only a minimal effort has been put into checking details.

Many thanks are due to the various people who participated; without their interest, we certainly wouldn't have made it even this far.

## 1. REVIEW FOR RINGS

1.1. **Revisionist Galois theory for fields.** Let  $L/K$  be an extension of fields, and  $G$  a finite group. Then  $L/K$  is  $G$ -Galois if  $G = \text{Aut}(L/K)$  and  $K = L^G$ . In this situation there is a bijection between subgroups  $H \leq G$  and intermediate fields  $L/F/K$ .

Here are two results from Galois theory for fields which find generalizations in the theory for rings:

- Hilbert's theorem 90:  $H^1(G, U(L)) = 0$ , where  $U(L) = L^\times$  denotes the group of units in  $L$ .
- $Br(K) = H^2(K^{sep}/K, U(K^{sep}))$ .

To prepare for the generalization to rings, we make a couple of definitions and reformulate the Galois condition.

**Definition 1.2** (Twisted Group Ring). As an  $L$ -vector space,  $L\langle G \rangle$  is  $L \times G$ . The multiplication is twisted by the action of  $G$ :

$$(x, g) \cdot (y, h) = (xg(y), gh).$$

Note that there is a natural map

$$j : L\langle G \rangle \rightarrow \text{Hom}_K(L, L)$$

given by

$$(x, g) \mapsto (t \mapsto x \cdot g(t)).$$

This is adjoint to the action of  $L$  on  $L\langle G \rangle$ :

$$L \otimes L\langle G \rangle \rightarrow L.$$

**Proposition 1.3.** *The map  $j$  is a  $K$ -algebra homomorphism and is bijective if and only if  $L/K$  is  $G$ -Galois.*

*Idea of proof.* Dedekind's lemma says that

$$\text{Alg}_K(A, L) \subset \text{Hom}_K(A, L)$$

is a linearly independent subset, so  $j$  is injective. When  $L/K$  is Galois,  $j$  is surjective by a dimension count. For details see [Dre95]. ...  $\square$

**Notation 1.4.** Let  $L^{\times G}$  denote  $\prod_G L$ , set maps from  $G$  to  $L$ .

---

*Date:* September 2010.

There is a natural map

$$h : L \otimes_K L \rightarrow L^{\times G}$$

given by

$$(x \otimes y) \mapsto (x \cdot g(y))_{g \in G}.$$

**Proposition 1.5.** *The map  $h$  is an  $L$ -algebra homomorphism, and is bijective if and only if  $L/K$  is  $G$ -Galois.*

*Idea of proof.* The map  $h$  is the  $L$ -module dual of  $j$ :

$$\mathrm{Hom}_L(L \otimes_K L, L) \cong \mathrm{Hom}_K(L, L)$$

and

$$\mathrm{Hom}_L(L^{\times G}, L) \cong L\langle G \rangle.$$

...  $\square$

**1.6. Basic definitions and theorems for rings.** Most of this subsection is based on Chapter 0 of Greither [Gre92].

The following theorem generalizes the previous results for fields; part *ii* is taken as the definition of  $G$ -Galois.

**Theorem 1.7.** *For a map (inclusion) of commutative rings  $R \rightarrow T$ , and a finite group  $G \leq \mathrm{Aut}_{R\text{-alg}}(T)$ , suppose  $R \cong T^G$ . Then the following are equivalent:*

- i.  $T$  is finitely-generated and projective over  $R$  and*  
 $j : T\langle G \rangle \rightarrow \mathrm{Hom}_R(T, T)$  *is an isomorphism (of  $R$ -algebras).*
- ii.  $h : T \otimes_R T \rightarrow T^{\times G}$  is an isomorphism (of  $T$ -algebras).*  
*(This implies that  $T$  is finitely-generated and projective over  $R$ .)*

*Remark 1.8.* Note that the Galois correspondence (Theorems 1.19 and 1.20) implies that in this case  $G = \mathrm{Aut}_{R\text{-alg}}(T)$ .

The following property is essential in many of the applications of Galois extensions:

**Lemma 1.9.** *If  $T/R$  is  $G$ -Galois, then  $T$  is faithfully flat over  $R$  ( $T \otimes_R -$  preserves and detects exact sequences).*

*Idea of proof.* Use Nakayama's lemma:  $T/\mathfrak{m}T \neq 0$  because  $T_{\mathfrak{m}} \neq 0$  ( $R_{\mathfrak{m}} \subset T_{\mathfrak{m}}$ ). ...  $\square$

The following results illustrate some of the benefits of faithful-flatness. Note however that this situation does not generalize to topological Galois extensions. Check the counterexamples of Wieland or Baker-Richter: Section 4.1, [BR10].

**Proposition 1.10.** *Let  $L$  be faithfully flat over  $R$ , and suppose that  $G$  is a group acting on  $T/R$ . If  $L \otimes_R T/L$  is  $G$ -Galois, then  $T/R$  is  $G$ -Galois.*

*Note.* It is important here that the  $G$ -action is induced by that on  $T/R$ ;  $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$  has a trivial  $\mathbb{Z}/3$  action, and hence extending scalars will always result in a trivial  $\mathbb{Z}/3$  action.

**Lemma 1.11.** *The trace map  $tr : T \rightarrow R$  splits  $R$  as a summand of  $T$*

*Note.* The previous result does not generalize to the topological setting.

**Lemma 1.12.** *Let  $T/R$  and  $T'/R$  be  $G$ -Galois extensions. Then every map of  $G$ -Galois extensions  $\varphi : T \rightarrow T'$  is an isomorphism.*

*Idea of proof.* Flat base change to the case of a trivial extension (which can always be done), and without loss of generality take  $R$  to be local. In that case, one can construct an orthogonal basis and check directly that  $\varphi$  is an isomorphism.  $\dots \square$

**Lemma 1.13.** *Let  $T/R$  be  $G$ -Galois. Then  $T$  is a finitely-generated projective  $R$ -module of rank 1 (an invertible module) over the group ring  $R[G]$ .*

*Idea of proof.*

$$T[G] \otimes_{R[G]} T \cong T \otimes_R T \cong T^{\times G}$$

Now  $T^{\times G}$  is invertible over  $T[G]$ , and hence we are done by faithfully flat descent.  $\dots \square$

1.14. **Galois correspondence.** The Galois correspondence for rings requires a couple of additional definitions; this could probably be explained more clearly using the language of Galois connections.

**Definition 1.15.** An  $R$ -algebra  $A$  is called *separable* over  $R$  if  $A$  is projective over  $A^e := A \otimes_R A^{op}$ .

**Definition 1.16.** Suppose  $G$  acts on  $T/R$ . Two elements  $\sigma, \tau \in G$  are called *strongly distinct* if for every non-zero idempotent  $e \in T$  there is some  $x \in T$  such that  $e \cdot \sigma(x) \neq e \cdot \tau(x)$ . Note that if  $T$  has no non-trivial idempotents then strongly distinct is the same as distinct.

**Definition 1.17.**  $T$  is called *connected* if it has no non-trivial idempotents

**Definition 1.18.** Suppose  $T/R$  is  $G$ -Galois and let  $U$  be an intermediate ring. We say that  $U$  is  $G$ -strong if the restrictions to  $U$  of any two elements of  $G$  are either strongly distinct or equal.

The following theorems come from the first chapter of [CHR65]. As done there, we put in brackets those statements which are vacuous in the case that  $T$  is connected.

**Theorem 1.19.** *For  $T/R$   $G$ -Galois and  $H \leq G$ , let  $U = T^H$ . Then*

- i.  $U$  is separable over  $R$  [and  $G$ -strong].*
- ii.  $T/U$  is  $H$ -Galois.*
- iii.  $H = \text{Aut}(T/U)$ .*
- iv. if  $H \triangleleft G$  then  $U/R$  is  $G/H$ -Galois.*

*Idea of proof.* Let  $\delta: T \rightarrow T^{\times G}$  be the  $T$ -module map induced by sending the unit of  $T$  to  $(\delta_{e\sigma})_{\sigma \in G}$ , the element which has a 1 in the  $e$ -coordinate and 0's elsewhere. The isomorphism  $h_{T/R}$  gives a lift in the diagram below.

$$\begin{array}{ccc}
 & & T \\
 & \swarrow & \downarrow \delta \\
 T \otimes_R T & \xrightarrow{h_{T/R}} & T^{\times G} \\
 \downarrow & \cong & \downarrow \\
 T \otimes_U T & \xrightarrow{h_{T/U}} & T^{\times H}
 \end{array}$$

Now the lift implies that  $h_{T/U}$  is surjective, which implies that  $j_{T/U}$  is an isomorphism and hence  $T/U$  is  $H$ -Galois. To see that  $U/R$  is separable, we have the following:  $T/U$  Galois implies that  $T$  is finitely-generated and projective over  $U \otimes_R U$ , which implies that  $T \otimes_R T$  is also finitely-generated and projective over  $U \otimes_R U$ .  $T/R$  Galois implies that  $T/R$  is separable, which is to say that  $T$  is finitely-generated and projective over  $T \otimes_R T$ . Now since  $T/U$  is Galois,  $U$  is a  $T$ -module summand of  $T$  and thus a  $U \otimes_R U$ -module summand of  $T$ . This implies that  $U$  is projective (and of course finitely-generated) over  $U \otimes_R U$ .

Let  $H' = \text{Aut}(T/U)$ . Then  $H \leq H'$  and  $T^H = U = T^{H'}$  so  $T/U$  is Galois for both  $H$  and  $H'$ . But the orders of  $H$  and  $H'$  give the rank of  $T \otimes_U T$ , and therefore  $H = H'$ .

If  $H \triangleleft G$ , in the case that  $T = R^{\times G}$ , one can see directly that  $T^H \cong T^{\times G/H}$ . By faithfully flat descent, this suffices.  $\dots \square$

**Theorem 1.20.** *Let  $T/R$  be  $G$ -Galois and  $U$  an intermediate ring which is separable over  $R$  [and  $G$ -strong]. Then there exists an  $H \leq G$  such that  $U = T^H$ .*

**Example 1.21.** For  $L/K$  a  $G$ -Galois extension of number fields,  $\mathcal{O}_L/\mathcal{O}_K$  is a  $G$ -Galois extension of rings if and only if  $L/K$  is unramified. For a maximal ideal  $\mathfrak{p}$ , one has a  $G$ -Galois extension  $(\mathcal{O}_L)_{\mathfrak{p}}/(\mathcal{O}_K)_{\mathfrak{p}}$  if and only if  $L/K$  does not ramify at  $\mathfrak{p}$ .

**Example 1.22.** To see that the  $G$ -strong condition is necessary, consider the following: Let  $T = \bigoplus_{i=1}^4 Re_i$  be a  $\mathbb{Z}/4$ -extension where the  $e_i$  are pairwise orthogonal idempotents with  $\sum e_i = 1$  and with the generator  $\sigma$  cyclicly permuting the  $e_i$ . Now let  $U = R(e_1 + e_2) \oplus R(e_3 + e_4)$ . Then  $\text{Aut}(T/U)$  is trivial, but the intermediate ring fixed by the trivial group is  $T$ , not  $U$ .

## 2. UNITS-PICARD-BRAUER FOR RINGS

The results of this section are contained in the AMS memoir of Chase–Harrison–Rosenberg [CHR65], mostly in the second chapter.

*Note.* Throughout this section,  $R$  will be a fixed commutative ring and  $\otimes$  will denote  $\otimes_R$ .

**2.1. The Brauer group.** Throughout this subsection, we consider a (noncommutative)  $R$ -algebra  $A$ . We begin with some definitions and a lemma:

### Definitions 2.2.

- $A$  is called *separable* over  $R$  if  $A$  is projective as a module over  $A^e$ . Since  $A$  is always finitely generated over  $A^e$ , this is equivalent to the condition that  $A$  be a dualizable module over  $A^e$ . By the dual basis lemma, this is equivalent to the condition that the coevaluation

$$A \otimes_{A^e} \text{Hom}_{A^e}(A, A^e) \rightarrow \text{Hom}_{A^e}(A, A)$$

be an isomorphism. To motivate this term, it should be noted that when  $R$  is a field,  $A$  being separable over  $R$  implies that  $A$  is semi-simple over  $R$  and remains semi-simple upon extension of scalars over any field extension of  $R$ . When  $A$  is also a field, this implies that  $A$  is a separable extension of  $R$  in the usual sense for fields [Coh03].

- $A$  is called *central* over  $R$  if the center of  $A$  is precisely  $R$ ; this occurs if and only if the unit

$$R \rightarrow \text{Hom}_{A^e}(A, A)$$

is an isomorphism.

- $A$  is called *faithfully projective* over  $R$  if both the coevaluation

$$\text{Hom}_R(A, R) \otimes A \rightarrow \text{Hom}_R(A, A)$$

and the evaluation

$$A \otimes_{A^e} \text{Hom}_R(A, R) \rightarrow R$$

are isomorphisms. Note, again by the dual basis lemma, that the coevaluation being an isomorphism is equivalent to  $A$  being finitely-generated and projective as an  $R$ -module. The evaluation map being an isomorphism implies that  $- \otimes_R A$  is faithful in the sense that  $M \otimes_R A = 0$  implies  $M = 0$ .

**Lemma 2.3** (See e.g. [KO74] or [DI71]). *If  $A$  is central and separable over  $R$ , then the evaluation*

$$\text{Hom}_{A^e}(A, A^e) \otimes_R A \rightarrow A^e$$

*is an isomorphism.*

An  $R$ -algebra  $A$  satisfying any (and therefore all) of the following conditions is called Azumaya over  $R$ . The Brauer group of  $R$ ,  $Br(R)$ , is the group of Morita equivalence classes of Azumaya  $R$ -algebras. The fact that this is a group follows from the definition of Azumaya.

**Theorem 2.4.** *The following are equivalent:*

- i.  $A$  is an invertible  $(R, A^e)$ -bimodule and thus  $A^e$  is Morita equivalent to  $R$ .
- ii.  $A$  is central and separable over  $R$ .
- iii.  $A$  is faithfully projective over  $R$  and the “sandwich map” (unit)

$$A^e \rightarrow \text{Hom}_R(A, A)$$

*is an isomorphism.*

- iv. There is an  $R$ -algebra  $B$  such that  $A \otimes_R B$  is Morita equivalent to  $R$ .

For an extension of commutative rings  $S/R$ , the relative Brauer group is the group of Azumaya  $R$ -algebras (up to Morita equivalence) which split upon extension to  $S$ :

$$Br(R, S) = Br(S/R) := \ker(Br(R) \xrightarrow{S \otimes_R -} Br(S)).$$

**2.5. Rigifying the Brauer group.** In this subsection we identify some particular Abelian groups covering  $Br(R, S)$ . This is motivated by the following:

**Lemma 2.6** (Splitting Lemma). *If  $A/R$  is Azumaya and  $S$  is the maximal commutative sub- $R$ -algebra of  $A$ , then  $S \otimes A^{op} \cong \text{End}_S(A)$  and hence  $A \in Br(R, S)$ .*

*Idea of proof.* First show that if  $h : B \rightarrow B'$  is a map of Azumaya  $R$ -algebras and  $\text{im}(h)$  contains its centralizer in  $B$ , then  $h$  is an isomorphism. Now  $S \otimes A^{op}$  is Azumaya over  $S$ , as is  $\text{End}_S(A)$ , and there is an obvious map  $S \otimes A^{op} \rightarrow \text{End}_S(A)$ . The centralizer of the image is the collection of endomorphisms given as left-multiplication by an element of  $S$ , and thus this map is an isomorphism. ... □

Now we describe a collection of Azumaya algebras which are split in a specific way. Let  $\mathcal{CAlg}_R$  denote the category of commutative  $R$ -algebras, and  $\mathcal{CAlg}_R^{f.p.}$  the full subcategory of commutative  $R$ -algebras which are faithfully projective as  $R$ -modules. Later, we will also make reference to the full subcategory of faithfully flat commutative  $R$ -algebras,  $\mathcal{CAlg}_R^{f.f.}$ . Let  $A(S, T)$  denote the following set:

$$A(S, T) := \{A \text{ Azumaya over } T \text{ with } i_A : S \otimes T \hookrightarrow A \text{ maximal commutative}\} / (\cong \text{ under } S \otimes T)$$

So  $A$  and  $B$  are equivalent in  $A(S, T)$  if there is an isomorphism of algebras commuting with the given inclusions of  $S \otimes T$ :

$$\begin{array}{ccc} & S \otimes T & \\ & \swarrow & \searrow \\ A & \xrightarrow{\cong} & B \end{array}$$

The sets  $A(S, T)$  can be made functorial: for a map  $(S, T) \rightarrow (S', T')$  in  $\mathcal{CAlg}_R^{f.p.} \times \mathcal{CAlg}_R$ , we define a map

$$\begin{aligned} A(S, T) &\rightarrow A(S', T') \text{ by} \\ A &\mapsto A' = \text{End}_A(S' \otimes_S A) \otimes_T T' \\ &\cong \text{End}_{A \otimes_T T'}(S' \otimes_S A \otimes_T T') \end{aligned}$$

where  $S' \otimes T' \rightarrow A'$  is defined by (left-mult.  $\otimes 1$ ). Note that  $A'$  and  $A \otimes_T T'$  represent the same class in  $Br(T')$  [CHR65, II.2.7].

Furthermore,  $A(S, T)$  has a naturally defined product: for  $A, B \in A(S, T)$ ,  $A \otimes_T B$  is an element of  $A(S \otimes S, T)$ , and the algebra structure of  $S$  induces a map  $A(S \otimes S, T) \rightarrow A(S, T)$ .

**Example 2.7.** Let  $J$  be an invertible module over  $S \otimes T$  (finitely-generated projective of rank 1). Then  $\text{End}_T(J)$  is an element of  $A(S, T)$ . In particular, let  $D = \text{End}_T(S \otimes T)$ .

**Proposition 2.8.** *Each  $A(S, T)$  is a commutative monoid with unit  $D = \text{End}_T(S \otimes T)$  and there is an exact sequence of commutative monoids*

$$\text{Pic}(R) \rightarrow \text{Pic}(S) \rightarrow A(S, R) \rightarrow \text{Br}(R, S) \rightarrow 0$$

*Idea of proof.*  $A(S, R)$  maps to  $\text{Br}(R, S)$  by the splitting lemma (2.6). ...  $\square$

**Corollary 2.9.** *By a diagram chase,  $A(S, R)$  is an abelian group.*

Now to close this subsection, we have something like a filtration for  $A(S, R)$  and  $\text{Pic}(S)$ :

**Definition 2.10.**

$$\begin{aligned} KA(S, T) &:= \ker(A(S, R) \rightarrow A(S, T)) \\ KPic(S, T) &:= \ker(\text{Pic}(S) \rightarrow \text{Pic}(S \otimes T)) \end{aligned}$$

**Proposition 2.11.** *For  $S \in \mathcal{CAlg}_R^{f.p.}$  we take colimits over  $T \in \mathcal{CAlg}_R$ : everything splits eventually.*

$$i. \text{Pic}(S) = \text{colim}_T KPic(S, T)$$

$$ii. A(S, R) = \text{colim}_T KA(S, T)$$

2.12. **Functors to  $\mathcal{Ab}$ .** The following functors to the category of abelian groups are of interest.

- The units functor  $U : \text{Alg}_R \rightarrow \mathcal{Ab}$
- The Picard functor  $\text{Pic} : \mathcal{CAlg}_R \rightarrow \mathcal{Ab}$
- The Brauer functor  $\text{Br}(R, -) : \mathcal{CAlg}_R \rightarrow \mathcal{Ab}$
- $A(-, -) : \mathcal{CAlg}_R^{f.p.} \times \mathcal{CAlg}_R \rightarrow \mathcal{Ab}$

In the previous subsection, we also saw  $KPic$  and  $KA$ . In this subsection we describe some general constructions for functors

$$F : \mathcal{CAlg}_R \rightarrow \mathcal{Ab}.$$

First, regard  $F$  as a functor of two variables by

$$F(S, T) = F(S \otimes T).$$

Now we make the following definitions:

$$\begin{aligned} (2.13) \quad QF(S, T) &= \text{coker}(F(T) \rightarrow F(S \otimes T)) \\ Q^n F(S, T) &= QF(S^{\otimes n-1}, T) \\ &= \text{coker}(F(S^{\otimes n-1} \otimes T) \rightarrow F(S^{\otimes n} \otimes T)) \end{aligned}$$

Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} F(T) & \longrightarrow & F(S \otimes T) & \longrightarrow & QF(S, T) \\ \downarrow & & \downarrow \varepsilon_1 & & \downarrow \Delta_1 \\ F(S \otimes T) & \xrightarrow{\varepsilon_0} & F(S^{\otimes 2} \otimes T) & \longrightarrow & Q^2 F(S, T) \\ \downarrow \varepsilon_0 - \varepsilon_1 & & \downarrow \varepsilon_1 - \varepsilon_2 & & \downarrow \Delta_2 \\ F(S^{\otimes 2} \otimes T) & \xrightarrow{\varepsilon_0} & F(S^{\otimes 3} \otimes T) & \longrightarrow & Q^3 F(S, T) \end{array}$$

The maps  $\varepsilon_i$  are induced by using the unit map  $R \rightarrow S$  in the  $i^{\text{th}}$  factor of  $S^{\otimes k} \rightarrow S^{\otimes k+1}$ . Note that the difference of the vertical and horizontal maps to  $F(S^{\otimes 3} \otimes T)$  is the differential in the Amitsur complex (defined below).

We end this subsection with one more definition:

$$(2.14) \quad KF(S, T) = \ker(\Delta_2).$$

Note that  $\Delta_2\Delta_1 = 0$ , so there is an induced natural transformation  $\Delta: QF \rightarrow KF$ .

*Remark 2.15* (Warning). Although the notation is similar to that for *KPic* and *KA*, these are defined in the previous subsection. These are the original notations in [CHR65], and we do not know if the similarity is meaningful.

Recall that  $D = \text{End}_R(S)$  is the unit of  $A(S, R)$ ;  $D \otimes T \cong \text{End}_T(S \otimes T)$  is the unit of  $A(S, T)$ .

**Lemma 2.16.**  $KU(S, T) \cong \text{Aut}_{A(S, T)}(D \otimes T)$ .

*Idea of proof.* For  $u \in U(S^{\otimes 2} \otimes T)$  representing a class in  $KU(S, T)$ , we have  $u$  acting on  $D \otimes T$  by conjugation. This correspondence gives the isomorphism. ...  $\square$

**Lemma 2.17.** For  $S \in \mathcal{CA}lg_R^{f.p.}$  and  $T \in \mathcal{CA}lg_S$ , the induced map

$$\Delta: QU(S, T) \rightarrow KU(S, T)$$

is an isomorphism.

*Idea of proof.* Use the multiplication  $S^{\otimes 3} \otimes T \xrightarrow{1 \otimes 1 \otimes \text{mult}} S^{\otimes 2} \otimes T$  to show that  $\Delta$  is surjective. ...  $\square$

### 3. THE AMITSUR COMPLEX FOR RINGS

*Note.* The content of this section comes from Chapter II of [CHR65], written by Chase and Rosenberg.

The Amitsur complex is a cosimplicial complex defined by

$$AC^\bullet(S/R) : [q] \mapsto S^{\otimes q+1}$$

with the usual cofaces and codegeneracies given by unit maps and multiplications. A cochain complex is constructed from this data by taking alternating sums of the coface maps.

If  $F$  is a functor to  $\mathcal{Ab}$ , we define  $AC^\bullet(S/R; F)$  by applying  $F$  termwise. The cohomology of the resulting complex is the Amitsur cohomology  $AH^*(S/R; F)$ .

**Lemma 3.1.** If  $f, f': T \rightarrow T'$  are commutative  $R$ -algebra maps, then  $f$  and  $f'$  induce chain homotopic maps on  $AC^\bullet(T/R; F)$ . Thus if  $T'$  is a commutative  $R$ -algebra retract of  $T$ , then  $AH^*(T'/R; F)$  is a retract of  $AH^*(T/R; F)$ .

*Remark 3.2.* The definition of the Amitsur complex makes sense for spectra and is discussed in [Rog08, §8.2]. For  $A \rightarrow B$  an extension of  $S$ -algebras ( $S$  the sphere spectrum here),  $A_B^\wedge := \text{Tot}(AC^\bullet(B/A))$  is a completion of  $A$  along the map to  $B$ , and this completion agrees with Bousfield  $B$ -nilpotent completion.

Moreover, for  $G$  acting on  $B/A$  we have  $h^\bullet: AC^\bullet(B/A) \rightarrow C^\bullet(G; B)$  induced by

$$h: B \wedge_A B \rightarrow F(G_+, B)$$

( $F$  the function spectrum here). If  $h$  is a weak equivalence, then  $h^\bullet$  is codegreewise a weak equivalence and induces a weak equivalence on  $\text{Tot}$ . We have  $\text{Tot}(C^\bullet(G; B)) = B^{hG}$  and thus the Galois question can be analyzed by considering the following:

$$\begin{array}{ccc} & & A_B^\wedge \\ & \nearrow & \downarrow \text{Tot}(h^\bullet) \\ A & \xrightarrow{i} & B^{hG} \end{array}$$

Now we return to the case of commutative rings.

**3.3. Identifying terms.** In this subsection, we consider  $AH^n(T/R; G_S)$  for functors  $G_S$  such as

- $F_S = F(S \otimes -)$ .
- $QF_S = QF(S, -)$ .
- $KF_S = KF(S, -)$ .

In our applications we will take  $F = U$  or  $F = Pic$ .

**Definition 3.4.** For a functor  $F$  to  $\mathcal{A}b$ , we define

$$AH^n(R; F) = \operatorname{colim}_T AH^n(T/R; F)$$

and

$$AHF^*(S) = \operatorname{colim}_T AH^*(T/R, F_S).$$

**Theorem 3.5.** *There is a natural transformation  $\psi : KA(S, T) \rightarrow AH^1(T/R; KU_S)$  which is an equivalence on  $\mathcal{C}Alg_R^{f.p.} \times \mathcal{C}Alg_R^{f.f.}$ .*

*Idea of proof.* Let  $A \in KA(S, T)$ , so  $S \otimes R \rightarrow A$  maximal commutative and we have

$$\begin{array}{ccc} & S \otimes T & \\ & \swarrow & \searrow \\ D \otimes T & \xrightarrow[\cong]{g} & A \otimes T \end{array}$$

(recall that  $D$  is the unit in  $A(S, R)$ ). We define  $\psi(A) = \bar{u} \in AC^1(T/R; KU_S)$  as follows: Let  $j$  be the composite

$$j : D \otimes T^{\otimes 2} \xrightarrow[\cong]{g} A \otimes T^{\otimes 2} \xrightarrow{1 \otimes \tau} A \otimes T^{\otimes 2} \xrightarrow[\cong]{g^{-1}} D \otimes T^{\otimes 2} \xrightarrow{1 \otimes \tau} D \otimes T^{\otimes 2}.$$

Then there is some  $u \in U(S^{\otimes 2} \otimes T^{\otimes 2})$  whose image  $\bar{u}$  in  $KU(S, T^{\otimes 2}) \subset Q^2U(S \otimes T^{\otimes 2})$  has the property that

$$j(x) = \bar{u} x \bar{u}^{-1}$$

(recall Lemma 2.16: automorphisms in  $A(S, T^{\otimes 2})$  are inner, by elements of  $KU(S, T^{\otimes 2})$ ). Check that  $\bar{u}$  is a cocycle. ... □

**Corollary 3.6.** *For each  $S \in \mathcal{C}Alg_R^{f.p.}$ , and each  $T \in \mathcal{C}Alg_S^{f.f.}$ , the transformation  $\psi$  above induces a natural isomorphism  $\psi : KA(S, T) \rightarrow AH^1(T/R; KU_S)$ .*

*Idea of proof.* Use the isomorphism of Lemma 2.17. ... □

**Theorem 3.7.** *There is a natural transformation  $\varphi : KPic(S, T) \rightarrow AH^1(T/R; U_S)$  which is an equivalence on  $\mathcal{C}Alg_R^{f.p.} \times \mathcal{C}Alg_R^{f.f.}$ .*

*Idea of proof.* Let  $J \in KPic(S, T)$ , so  $J \in Pic(S)$  and we have

$$S \otimes T \xrightarrow[\cong]{g} J \otimes T.$$

We define  $\varphi(J) = u \in U(S \otimes T^{\otimes 2}) = AC^1(T/R; U_S)$  as follows: Let  $j$  be the composite

$$j : S \otimes T^{\otimes 2} \xrightarrow[\cong]{g} J \otimes T^{\otimes 2} \xrightarrow{1 \otimes \tau} J \otimes T^{\otimes 2} \xrightarrow[\cong]{g^{-1}} S \otimes T^{\otimes 2} \xrightarrow{1 \otimes \tau} S \otimes T^{\otimes 2}.$$

Then there is some  $u \in U(S \otimes T^{\otimes 2})$  such that

$$j(x) = ux.$$

Check that  $u$  is a cocycle. ... □



**Lemma 3.8.** *The following comparison diagram commutes for  $S \in \mathcal{CAlg}_R^{f.p.}$  and  $T \in \mathcal{CAlg}_S^{f.f.}$ .*

$$\begin{array}{ccc} KPic(S, T) & \longrightarrow & KA(S, T) \\ \downarrow \varphi & & \downarrow \psi \\ AH^1(T/R; U_S) & \xrightarrow{AH^1(proj.)} & AH^1(T/R; QU_S) \end{array}$$

*Remark 3.9.* [CHR65] Makes reference to an alternative technique for a unified proof of the previous two theorems, using various fibered categories over  $\mathcal{CAlg}_R^{f.p.} \times \mathcal{CAlg}_R$  and analyzing the distinguished object  $X_0$  of each. In each case one has

$$Aut(X_0)(T) \cong Aut(X_0 \otimes T)$$

and a comparison with  $H^n(T/R, Aut(X_0))$ .

**Theorem 3.10.**

*i. The following sequence is exact.*

$$\begin{array}{ccccccccc} Pic(R) & \longrightarrow & Pic(S) & \longrightarrow & A(S, R) & \longrightarrow & Br(R, S) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ AHU^1(R) & \longrightarrow & AHU^1(S) & \longrightarrow & AHQU^1(S) & \longrightarrow & AHU^2(R) & \longrightarrow & AHU^2(S) \end{array}$$

*ii. As functors of  $S \in \mathcal{CAlg}_R^{f.p.}$ , we have natural transformations*

$$\begin{array}{ccccccccc} U(S) & & KPic(S, T) & \longrightarrow & Pic(S) & & KA(S, T) & \longrightarrow & A(S, R) & & Br(R, S) \\ \downarrow \cong & & \downarrow \varphi_T & & \downarrow \text{colim } \varphi_T & & \downarrow \psi_T & & \downarrow \text{colim } \psi_T & & \downarrow \\ AHU^0(S) & & AH^1(T/R; U_S) & \longrightarrow & AHU^1(S) & & AH^1(T/R; QU_S) & \longrightarrow & AHQU^1(S) & & AHU^2(R) \end{array}$$

*iii. The following sequence is exact.*

$$0 \rightarrow Br(R, S) \rightarrow AHU^2(R) \rightarrow AHU^2(S)$$

*Idea of proof.*

*i.* Exactness of the top row follows from the definition of  $A(S, T)$ . Exactness of the bottom row follows from the long exact sequence in homology induced by the short exact sequence

$$0 \rightarrow U \rightarrow U_S \rightarrow QU(S, -) \rightarrow 0$$

for  $S \in \mathcal{CAlg}_R^{f.p.}$ .

*ii.*  $AHU^2(R)$  is a constant functor of  $S$ , and the dashed map follows from existence of the others by exactness of part (i).

... □

**3.11. Spectral sequence.** First we introduce the bigraded Amitsur complex:

$$C^{p,q}(S, T/R; F) = F(S^{\otimes p+1} \otimes T^{\otimes q+1}).$$

Then we have a spectral sequence

$$AH^p(S/R; AH_T F^q) \Rightarrow H^{p+q} C^\bullet(S, T/R, F)$$

where  $AH_T F^q(S) = AH^q(T/R; F_S)$ .

**Lemma 3.12.** *If  $T$  is an  $S$ -algebra, then*

$$H^{p+q} C^\bullet(S, T/R, F) \cong AH^{p+q}(T/R; F).$$

*Idea of proof.* This follows from Lemma 3.1.

... □

Taking the colimit defining  $AH^*(R; F)$  over  $S$ -algebras  $T$ , we have

$$(3.13) \quad AH^p(S/R; AHF^q) \Rightarrow AH^{p+q}(R; F).$$

**Lemma 3.14.** *Let  $E_r^{p,q} \Rightarrow H^{p+q}$  be a first-quadrant spectral sequence. Then there is a 7-term exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow F^1 H^2 \longrightarrow E_2^{1,1} \longrightarrow E_2^{3,0}$$

where  $F^1 H^2$  is the first filtration group.

**Corollary 3.15.** *Taking  $F = U$  in the spectral sequences above, we have*

$$\begin{aligned} 0 \longrightarrow AH^1(S/R; U) \longrightarrow Pic(R) \longrightarrow AH^0(S/R; Pic) \longrightarrow AH^2(S/R; U) \longrightarrow \\ \longrightarrow Br(R, S) \longrightarrow AH^1(S/R; Pic) \longrightarrow AH^3(S/R; U). \end{aligned}$$

*Idea of proof.* To identify the first filtration group as  $Br(R, S)$ , note that the inclusion  $r: T \rightarrow S \otimes T$  induces a commuting map

$$\begin{array}{ccc} AH^*(T/R; F) & & \\ \downarrow \eta & \searrow r & \\ H^*C^\bullet(S, T/R, F) & \longrightarrow & AH^*(T/R; F_S). \end{array}$$

If  $T$  is an  $S$ -algebra, then  $\eta$  is an isomorphism and hence  $\ker(r) \cong F^1 AH^*(T/R; F_S)$ . Passing to colimits over  $S$ -algebras  $T$  gives  $F^1 AH^n(R; F) = \ker(AH^n(R; F) \rightarrow AH^n(R; F_S))$ . By Theorem 3.10 (iii), this kernel is  $Br(R, S)$ . ...  $\square$

### 3.16. Galois extensions.

**Theorem 3.17** ([CHR65, Theorem I.5.4]). *For  $S/R$   $G$ -Galois, we have*

$$\begin{aligned} H^*(S/R; U) &\cong H^*(G; U(S)) \\ H^*(S/R; Pic) &\cong H^*(G; Pic(S)). \end{aligned}$$

## 4. HIGHLY STRUCTURED RING SPECTRA

Spectra are representing objects for generalized cohomology theories on based spaces. Naturally isomorphic cohomology theories correspond to homotopy equivalent spectra, and cohomology theories with product structures correspond to ring spectra. To develop the Galois theory of ring spectra, it is necessary to work in a category of highly-structured spectra: a closed monoidal category with a notion of homotopy such that its homotopy category is equivalent to the category of spectra (cohomology theories). Monoid objects in such a category are highly-structured ring spectra, and there is a natural theory of module spectra over these ring spectra, etc. There are a number of different constructions which give categories of highly-structured spectra, but we will not say more about that here.

### 4.1. Non-Faithfulness of Galois Extensions.

**Definition 4.2.** An extension of rings  $A \rightarrow B$  is faithful if, for any  $A$ -module  $M$ ,

$$M \wedge_A B \simeq * \quad \Rightarrow \quad M \simeq *$$

The following example is due to Ben Wieland, although the fact that this is a Galois extension, and the proposition with which to prove it is non-faithful, are both contained in [Rog08].

**Proposition 4.3.** *A Galois extension  $A \rightarrow B$  is faithful if and only if the topological norm below is an equivalence.*

$$(B \wedge S^{adG})_{hG} \xrightarrow[\simeq]{Norm} B^{hG} \longrightarrow B^{tG}$$

*Idea of proof.* If  $Norm$  is an equivalence, let  $M$  be such that  $M \wedge_A B \simeq *$ . Then

$$M \wedge_A (B \wedge S^{adG})_{hG} \xrightarrow[\simeq]{Norm} M \wedge_A B^{hG} \simeq M \wedge_A A.$$

But smashing with  $M$  commutes with homotopy orbits, so the first term is

$$M \wedge_A (B \wedge S^{adG}) \simeq (M \wedge_A B \wedge S^{adG}) \simeq *.$$

Now suppose  $A \rightarrow B$  is faithful; we see that  $B \wedge_A Norm$  is an equivalence: We have

$$\begin{array}{ccc} B \wedge_A (B \wedge S^{adG})_{hG} & \xrightarrow{B \wedge_A Norm} & B \wedge_A B^{hG} \\ \left| \simeq \right. & & \left| \simeq \right. \\ (B \wedge_A B \wedge S^{adG})_{hG} & & B \\ \left| \simeq \right. & & \parallel \\ (Map(G_+, B) \wedge S^{adG})_{hG} & \xrightarrow{(1)} & Map(G_+, B)^{hG} \end{array}$$

and the map (1) is always an equivalence ...  $\square$

**Example 4.4.**  $EG \rightarrow BG$  induces

$$Map(BG_+, H\mathbb{F}_p) \rightarrow Map(EG_+, H\mathbb{F}_p) \simeq H\mathbb{F}_p.$$

and ([Rog08]) this extension is  $G$ -Galois if and only if  $G$  acts nilpotently on  $\mathbb{F}_p[G]$  (e.g. if  $G$  is a finite group). The Tate spectrum in this case has

$$\pi_* H\mathbb{F}_p^{tG} = \hat{H}^{-*}(G; \mathbb{F}_p) \neq 0$$

where  $\hat{H}$  is Tate cohomology and the first equality holds because  $G$  acts trivially on  $H\mathbb{F}_p \simeq Map(EG_+, H\mathbb{F}_p)$ .

Baker and Richter [BR10] have shown that extensions

$$E_n^{BG} = Map(BG_+, E_n) \rightarrow Map(EG_+, E_n) \simeq E_n$$

and

$$Map(BG_+, K(n)) \rightarrow Map(EG_+, K(n)) \simeq K(n)$$

are faithful extensions, although the second is a non-commutative extension and is not Galois (in the associative setting) and the first fails to be Galois in many cases.

## 5. MORAVA $K$ -THEORIES AND LUBIN-TATE SPECTRA

**5.1. Motivation.** The purpose of this subsection is to give some motivating facts about the deep connection between formal group theory and calculations in stable homotopy. It is unlikely that it will get typed up. The main ideas were the following; taken mostly from [Rav92]. Jacob Lurie has a collection of course notes [Lur10] which are quite nice.

**Definition 5.2.** A self-map  $f: \Sigma^d X \rightarrow X$  can be composed with appropriate suspensions of itself. A self-map is nilpotent if some iterate is null-homotopic; otherwise it is periodic.

**Theorem 5.3** (Nilpotence).  *$MU_*$  detects nilpotence.*

**Corollary 5.4.**  *$\pi_* S$  consists entirely of nilpotent elements for  $* > 0$ .*

**Theorem 5.5.** *Description/characterization of Morava  $K(n)$ : generalizations of mod  $p$  complex  $K$ -theory for height  $n$ .  $K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}]$  with  $|v_n| = 2(p^n - 1)$ .*

**Definition 5.6.**  $X$  has type  $n$  if  $n$  is minimal such that  $K(n)_*(X) \neq 0$ . If  $X \simeq pt.$ , then  $X$  has type  $\infty$ .

**Theorem 5.7** (Periodicity). *Spaces of finite type  $n$  have periodic self-maps which are isomorphisms on  $K(n)_*$ , and these have a certain asymptotic uniqueness. Such a map is called a  $v_n$ -map (think of multiplication by  $v_n$ ).*

*Remark 5.8.* Periodic self maps give rise to infinite families of non-trivial elements in  $\pi_*S$ . These are constructed by looking at bottom-cell inclusion and top-cell collapse on the space involved. Early examples were constructed by Toda, Mahowald, Adams. These may be the well-known “greek letter elements”.

### 5.9. Bousfield classes and chromatic convergence.

**Definition 5.10.** A thick subcategory contains the zero object and is closed under fibers/cofibers and retracts.

**Example 5.11.** The subcategory  $\mathcal{T}_{\geq n}$  of finite  $p$ -local CW complexes of type  $\geq n$  is a thick subcategory of  $\mathcal{T}$ , the category of finite  $p$ -local CW complexes.

**Theorem 5.12** (Thick Subcategory). *Every thick subcategory of  $\mathcal{T}$  is one of the  $\mathcal{T}_{\geq n}$ .*

*Remark 5.13.* This also gives a classification of those thick subcategories of  $p$ -local spectra which are generated by  $p$ -local finite spectra under colimits.

**Definition 5.14** (Bousfield classes).  $\langle E \rangle = \langle F \rangle$  if  $L_E \simeq L_F$ .

**Proposition 5.15.**

$$\langle E(n) \rangle = \langle E_n \rangle = \bigvee_{i=0}^n \langle K(n) \rangle$$

**Theorem 5.16** (Chromatic convergence). *Localization with respect to  $E(n)$  gives a tower of approximations, and the (homotopy?-)limit of this tower is the identity functor.*

### 5.17. Formal group laws.

**Definition 5.18.** A formal group over a ring  $k$  is a connected, commutative, cocommutative, 1-dimensional, topological Hopf algebra  $\mathcal{A}$  over  $k$ . This collection of adjectives implies that  $\mathcal{A} \cong k[[x]]$ .

A formal group law over  $k$  is a specified isomorphism (called a coordinate)  $k[[x]] \xrightarrow{\cong} \mathcal{A}$ . The Hopf algebra structure determines a power series in two variables,  $F(x, y) \in k[[x, y]]$  which satisfies commutativity, associativity, and a unit condition. Such a power series is called a formal group law.

**Theorem 5.19.** *There is a universal formal group law  $U$  over the Lazard ring,  $L$ , and  $(L, U)$  corepresents the formal-group-laws functor.*

**Theorem 5.20.** *The spectrum  $MU$  is a “topological lift” of  $L$ :  $MU_* \cong L$ , and  $MU_*$  carries a natural formal group law induced by the multiplication on  $\mathbb{C}P^\infty$ . This natural formal group law is the universal formal group law.*

**Proposition 5.21.** *For any formal group law  $F$ , we have*

$$[p]_F(x) \equiv_{(\text{mod } p)} a \cdot x^{p^n} + \dots$$

*for some coefficient  $a$  and some number  $n$ .*

**Definition 5.22.** This value of  $n$  is called the height of  $F$ .

*Remark 5.23.* The height of a formal group law and the type of a spectrum are closely related quantities. The coefficient of  $x^{p^n}$  in  $p$ -series of the universal formal group law is closely related to the  $v_n$  of the Morava  $K(n)$ .

**Theorem 5.24.** *Over  $\overline{\mathbb{F}}_p$ , two formal group laws are isomorphic if and only if they have the same height.*

The Lubin-Tate ring is

$$(5.25) \quad (E_n)_0 = W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]]$$

with  $|u_i| = 0$ . This ring carries a formal group law which is the universal deformation of the Honda formal group law (the unique formal group law over  $\mathbb{F}_{p^n}$  whose  $p$ -series mod  $p$  is precisely  $x^{p^n}$ ).

**Theorem 5.26** (Hopkins-Miller). *There is a highly-structured ring spectrum  $E_n$  and a map of highly-structured ring spectra  $MU \rightarrow E_n$  which induces the formal group law described above on*

$$(E_n)_* = W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]] \quad |u| = 2.$$

*The spectrum  $E_n$  has an action of the extended Morava stabilizer group  $\mathbb{G}_n = \mathbb{S}_n \rtimes C_n$  through maps of highly-structured spectra.*

5.27. **Deformations.** This section follows [Rez98].

**Definition 5.28** (Deformations). Let  $(k, \Gamma)$  be a field of characteristic  $p$  and a formal group law over the field. A deformation of  $(k, \Gamma)$  to a complete local ring  $B$  is  $(G, i)$  where  $G$  is a formal group law over  $B$  and  $i$  is an inclusion

$$k \xrightarrow{i} B/\mathfrak{m}$$

such that  $i^*\Gamma = \pi^*G$ . ( $\pi : B \rightarrow B/\mathfrak{m}$ .) A morphism

$$(i_1, G_1) \rightarrow (i_2, G_2)$$

is determined by the following:  $i_1 = i_2$  and an isomorphism of formal group laws  $f : G_1 \rightarrow G_2$  such that  $\pi^*f$  is the identity. This is sometimes called a  $\star$ -isomorphism.

Here is a picture of the groupoid of deformations of  $(k, \Gamma)$  to  $B$ . It is a pull-back square of groupoids:

$$\begin{array}{ccc} Def_{(k, \Gamma)}(B) & \longrightarrow & fgl(B) \\ \downarrow & & \downarrow \\ \coprod_{i:k \rightarrow B/\mathfrak{m}} \{\Gamma\} & \longrightarrow & \coprod_{i:k \rightarrow B/\mathfrak{m}} fgl(k) \longrightarrow fgl(B/\mathfrak{m}) \end{array}$$

**Theorem 5.29** (Lubin-Tate). *If  $\Gamma$  is a formal group law of height  $n$  over  $k$  (field of characteristic  $p$ ), and  $(i, G)$  is a deformation complete local ring  $B$ , then*

$$\begin{aligned} \pi_1(Def_{k, \Gamma}(B)_i, (G, i)) &= \{1\} \\ \pi_0(Def_{(k, \Gamma)}(B)_i) &= \mathfrak{m}^{\times(n-1)} \end{aligned}$$

where  $Def_{(k, \Gamma)}(B)_i$  denotes deformations of the form  $(G, i)$  for  $i$  fixed.

Furthermore, there is a complete local ring  $E(k, \Gamma)$  and an isomorphism

$$k \xrightarrow{\cong} E(k, \Gamma)/\mathfrak{m}$$

with a formal group law  $F_{(k, \Gamma)}$  over  $E(k, \Gamma)$  such that  $(F_{(k, \Gamma)}, id)$  is a universal deformation (described below).

*Remark 5.30.*  $\pi_0$  denotes the set of connected components, and  $\pi_1$  is the automorphism group of the base point.

Saying that  $(F_{(k,\Gamma)}, id)$  is a universal deformation means that for any complete local ring  $B$  there is a canonical map  $i_E : E(k, \Gamma) \rightarrow B$  and  $E(k, \Gamma)$  together with its formal group law corepresents the functor

$$B \mapsto \pi_0(Def_{k,\Gamma})(B)_{i_E}.$$

Here is a picture:

$$\begin{array}{ccc} E(k, \Gamma) & \xrightarrow{\exists! \phi} & B \\ \downarrow & & \downarrow \\ k & \xrightarrow{i_E} & B/\mathfrak{m} \end{array} \quad \begin{array}{ccccc} F & \dashrightarrow & \phi^* F & \longleftarrow & G \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \dashrightarrow & i^* \Gamma & \longleftarrow & \pi^* G \end{array}$$

When  $k = \mathbb{F}_{p^n}$  and  $\Gamma$  is the Honda formal group law ( $[p](x) \equiv_{\text{mod } p} x^{p^n}$ ), then

$$E(\mathbb{F}_{p^n}, \text{Honda}_n) = W\mathbb{F}_{p^n} \llbracket u_1, \dots, u_{n-1} \rrbracket = (E_n)_0$$

**Definition 5.31** (Witt vectors).  $W\mathbb{F}_{p^n}$  denotes the Witt vectors of  $\mathbb{F}_{p^n}$ . As ring, this is  $\mathbb{Z}_p[\zeta]$  where  $\zeta$  is a primitive  $(p-1)^{\text{st}}$  root of 1. It is the ring of integers in the Galois extension  $\mathbb{Q}_p[\zeta]/\mathbb{Q}_p$ .

**Definition 5.32** (Morava stabilizer group).  $\mathbb{S}_n$  is the group of formal group law automorphisms of the height- $n$  Honda formal group law over  $\mathbb{F}_{p^n}$ .

*Note.* The Johnson-Wilson spectrum,  $E(n)$  is closely related to  $E_n$ ;  $E(n)_* = \mathbb{F}_{p^n} \llbracket u_1, \dots, u_{n-1} \rrbracket [u^{\pm 1}]$

### 5.33. Galois extensions.

**Theorem 5.34** (Devinatz-Hopkins [Rog08]).

i. For closed subgroups  $H \subset K \subset \mathbb{G}_n = \mathbb{S}_n \times C_n$  with  $H \triangleleft K$  and  $[K : H] < \infty$ ,

$$E_n^{hK} \rightarrow E_n^{hH}$$

is a  $K(n)$ -local  $K/H$ -Galois extension.

ii. In particular, if  $K$  is finite,

$$E_n^{hK} \rightarrow E_n$$

is  $K(n)$ -local  $K$ -Galois.

iii. If  $U$  is an open normal subgroup of  $\mathbb{G}_n$  (hence of finite index), then

$$L_{K(n)} S \rightarrow E_n^{hU}$$

is  $K(n)$ -local  $\mathbb{G}_n/U$ -Galois. ( $E_n^{h\mathbb{G}_n} \simeq L_{K(n)} S$ )

iv. Taking a descending sequence of open normal subgroups whose intersection is trivial, we have a  $K(n)$ -local profinite Galois extension with group  $\mathbb{G}_n$ :

$$L_{K(n)} S \rightarrow E_n.$$

### REFERENCES

- [BR10] A. Baker and B. Richter, *Some properties of Lubin-Tate cohomology for classifying spaces of finite groups*, Arxiv preprint arXiv:1005.1662, 2010.
- [CHR65] S. U. Chase, D. K. Harrison, and Alex Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. No. **52** (1965), 15–33. MR MR0195922 (33 #4118)
- [Coh03] P. M. Cohn, *Further algebra and applications*, Springer-Verlag London Ltd., London, 2003.
- [DI71] Frank DeMeyer and Edward Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics, Vol. 181, Springer-Verlag, Berlin, 1971. MR MR0280479 (43 #6199)
- [Dre95] A.W.M. Dress, *One more shortcut to Galois theory*, Advances in Mathematics **110** (1995), no. 1, 129–140.
- [Gre92] C. Greither, *Cyclic Galois extensions of commutative rings*, Lecture Notes in Math, vol. 1534, Springer-Verlag, Berlin-Heidelberg-Tokyo-New York, 1992.

- [KO74] M.A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Math, vol. 389, Springer-Verlag, Berlin-Heidelberg-Tokyo-New York, 1974.
- [Lur10] J. Lurie, *Chromatic Homotopy Theory Course Notes*, electronic, 2010, <http://www.math.harvard.edu/~lurie/252x.html>.
- [Rav92] D.C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Princeton Univ Pr, 1992.
- [Rez98] C. Rezk, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations: proceedings of a Conference on Homotopy Theory, March 23-27, 1997, Northwestern University, Amer Mathematical Society, 1998, p. 313.
- [Rog08] John Rognes, *Galois extensions of structured ring spectra. Stably dualizable groups.*, Mem. Amer. Math. Soc. **192** (2008), no. 898, viii+137. MR MR2387923 (2009c:55007)

UNIVERSITY OF GEORGIA, DEPARTMENT OF MATHEMATICS, ATHENS, GA 30606