Algebraic K-theory for 2-categories

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Outline

- Algebraic K-groups of a ring
  - Low-dimensional cases
  - Quillen's higher K-groups
  - Application for K_2
- 2-categorical analogue
  - Examples of symmetric monoidal 2-categories
  - Generalizations of Quillen's constructions
  - Application for K_3

Algebraic K-groups of a ring

Let R be a ring.

Let $\mathrm{Proj}_{f.g.}(R)$ be the category of finitely-generated and projective R-modules.
This category is symmetric monoidal under $\oplus$.

Gr denotes group-completion of a monoid
(adjoin formal inverses, like localization of a ring).
Also known as Grothendieck group.

Algebraic K-groups of a ring: $K_0(R)$

Definition

$$K_0(R) = Gr(\mathrm{Proj}_{f.g.}(R)/\cong).$$

Example for field F, $K_0(F) = \mathbb{Z}$.

Example Application $\det\colon K_0(R) \to \mathrm{Pic}(R)$ is a surjection.

Generalizes to Grothendieck group of other symmetric monoidal categories:

- vector bundles on topological spaces
- representations of finite groups

Algebraic K-groups of a ring: $K_1(R)$

Let $GL(R)$ be the infinite general linear group

$$GL(R) = \operatorname{colim}\Big(\cdots \hookrightarrow GL_n(R) \hookrightarrow GL_{n+1}(R) \hookrightarrow \cdots \Big).$$

Definition

$$K_1(R) = GL(R) \Big/ [GL(R), GL(R)].$$

Equivalently,

$$K_1(R) = H_1(GL(R)).$$

Algebraic K-groups of a ring: $K_1(R)$

Theorem

Let S be a multiplicatively closed set of central elements in R. Then there is an exact sequence

$$K_1(R) \to K_1(S^{-1}R) \to K_0(\mathbf{Ft}_S(R)) \to K_0(R) \to K_0(S^{-1}R).$$

Note: $\mathbf{Ft}_S(R)$ is the category of S-torsion R-modules with finite-length projective resolutions.

Theorem (Fundamental Theorem)

There is an exact sequence

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t, t^{-1}]) \to K_0(R) \to 0.$$

Algebraic K-groups of a ring: $K_2(R)$

The elementary group $E_n(R)$ is the group generated by elementary $n \times n$ matrices.
(ones on diagonal; single off-diagonal entry)

$E(R) = \mathrm{colim}_n E_n(R)$.

Lemma (Whitehead)

$$E(R) = [GL(R), GL(R)].$$

The Steinberg group, $St(R)$ is generated by formal symbols $x_{ij}(r)$ for $r \in R$, subject to elementary relations.
(products and commutators of elementary matrices)

Algebraic K-groups of a ring: $K_2(R)$

By definition we have a surjection $St(R) \to E(R)$, but $E(R)$ generally has more relations than $St(R)$.

Definition

$$K_2(R) = \ker\big(St(R) \to E(R)\big).$$

Theorem (Steinberg)

$K_2(R)$ is the center of $St(R)$. In particular, $K_2(R)$ is abelian.

Theorem (Bass)

$$K_2(R) \cong H_2(E(R)).$$

Properties of low-dimensional K-groups

Looking at the definitions, it may be unclear that the groups $K_i(R)$ are part of any reasonable sequence. Here are some clues.

- Localization exact sequence
- Fundamental Theorems for $R[t, t^{-1}]$
- $K_1(R)$ and $K_2(R)$ are modules over $K_0(R)$
- Product $K_1(R) \otimes K_1(R) \to K_2(R)$.

Quillen's higher K-groups

Definition

$$K_n(R) = \pi_n\big(BGL(R)^+\big), \quad n > 0$$

Explanation (Part 1) BG is the classifying space of a group G.

It is the base of a principal G-bundle

$$G \to EG \to BG$$

with total space EG, a contractible space with free G-action. So $BG \simeq K(G, 1)$ is an Eilenberg-Mac Lane space of type $(G, 1)$.

Example

$$\mathbb{Z} \to \mathbb{R} \to S^1$$

Quillen's higher K-groups

Definition

$$K_n(R) = \pi_n\big(BGL(R)^+\big), \quad n > 0$$

Explanation (Part 2) The plus construction X^+ on a topological space X has two properties:

- $\pi_1(X^+) = \pi_1(X)^{ab}$
- a map $X \to X^+$ inducing a homology isomorphism

Thus we certainly have

$$\pi_1(BGL(R)^+) \cong K_1(R).$$

Quillen's higher K-groups

Definition

$$K_n(R) = \pi_n\left(BGL(R)^+ \right), \quad n > 0$$

Explanation (Part 3) Why is this definition such a good one? (partial answer)

- Extends localization exact sequence and fundamental theorem.
- Explains graded ring structure on $K_n(R)$.
- Explains connection between algebraic K-groups and homotopy theory.

Quillen's higher K-groups: Second version

Let $\mathbb{Q} = \mathbb{Q}(\mathrm{Proj}_{f.g.}(R))$ be the following category.

- Objects are those of $\mathrm{Proj}_{f.g.}(R)$
- Morphisms $X \to Y$ are injection/surjection spans

$$\begin{array}{ccccc} & & S & & \\ & \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow & \\ X & & & & Y \end{array}$$

- Composition is via pullback

$$\begin{array}{ccccccc} & & & P & & & \\ & & \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow & & \\ & S & & & & T & \\ \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow & & \swarrow\!\!\swarrow & & \hookrightarrow\!\!\searrow \\ X & & & Y & & & Z \end{array}$$

Quillen's higher K-groups: Second version

Definition (Alternate)

$$K_n(R) = \pi_{n+1}(B\mathrm{Q}) \qquad n \geq 0$$

Explanation

- $B\mathrm{Q}$ is classifying space of a category
- $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ for any X
- $\Omega B\mathrm{Q}$ is a topological group-completion
  - group-completion on π_0
  - isomorphism on homology when π_0 inverted

The Q-construction is purely algebraic; doesn't rely on plus construction. Iterated Q-construction (Waldhausen's $S_\bullet$) has further structure.

Quillen's higher K-groups: $+ = \mathbf{Q}$

Theorem (Quillen)

$$K_0(R) \times BGL(R)^+ \simeq \Omega B\mathbf{Q}.$$

This gives two completely different ways to approach algebraic K-theory. Each version has both conceptual and calculational advantages.

Proof sketch: $+ = S^{-1}S = \mathbf{Q}$

Let $S = \mathrm{Proj}^{\mathrm{iso}}_{f.g.}(R)$. Define localization of categories, $S^{-1}S$, and prove

$$K_0(R) \times BGL(R)^+ \simeq BS^{-1}S \simeq \Omega B\mathbf{Q}.$$

(We will say more about $S^{-1}S$, but not $\mathbf{Q}$.)

$S^{-1}S$

Let $S = (S, \otimes)$ be a symmetric monoidal category with

- every morphism invertible (S is a groupoid);
- faithful translations ($X \otimes A \cong Y \otimes A$ iff $X \cong Y$).

Define new category $S^{-1}S$ with formal inverses to $\otimes$.

- Objects (X_1, X_2) pairs of objects from S (formal fractions).
- Morphisms $(X_1, X_2) \to (Y_1, Y_2)$ equivalence classes of triples (A, f_1, f_2) where $A \in S$ and f_1, f_2 morphisms in S and $(f_1, f_2)\colon (X_1 \otimes A, X_2 \otimes A) \to (Y_1, Y_2)$.
- Equivalence relation $(A, f_1, f_2) \sim (A', f_1', f_2')$ whenever $\exists\, t\colon A \to A'$ and $f_i = \left(X_i \otimes A \xrightarrow{1 \otimes t} X_i \otimes A' \xrightarrow{f_i'} Y_i \right)$ in S.

$S^{-1}S$

Examples

- $S = \mathrm{Proj}^{iso}_{f.g.}(R)$ with $\oplus$
- $S = \coprod_n \Sigma_n \simeq$ FinSet with (block) sum / disjoint union.

Theorem (Quillen, Grayson)

The inclusion $S \to S^{-1}S$ induces a topological group-completion.

That means the map on classifying spaces

$$BS \to BS^{-1}S,$$

- induces group-completion on π_0;
- is a homology localization: $H_*BS^{-1}S \cong H_*(BS)[\pi_0(BS)^{-1}]$.

$S^{-1}S$

Generalization of $GL(R)$: Given a sequence of objects and inclusions

$$\cdots \hookrightarrow \mathrm{Aut}_S(A_n) \hookrightarrow \mathrm{Aut}_S(A_{n+1}) \hookrightarrow \cdots,$$

we have a group $\mathrm{Aut}(S) = \mathrm{colim}_n \mathrm{Aut}_S(A_n)$.
(This is deceptively simple.)

If (as in our examples), the sequence is cofinal in S, one proves

$$BS \to K_0(S) \times B\mathrm{Aut}(S)$$

is an isomorphism on localized homology $H_*(\,\_\_\,)[\pi_0(BS)^{-1}]$.
(This is another topological group-completion.)

$S^{-1}S$

Therefore the solid arrows are isomorphisms on $H_*(\_\_)[\pi_0(BS)^{-1}]$

$$\begin{array}{ccc} BS & \longrightarrow & K_0(S) \times B\mathrm{Aut}(S) \\ \downarrow & & \downarrow \\ BS^{-1}S & \dashrightarrow & K_0(S) \times B\mathrm{Aut}(S)^+ \end{array}$$

Therefore by universal property we have the dashed arrow, which induces a homology isomorphism. Since these are simple spaces, this induces a homotopy equivalence

$$BS^{-1}S \simeq K_0(S) \times B\mathrm{Aut}(S)^+.$$

This is half of the $+ = \mathrm{Q}$ theorem.

Example: $K_2(R) \cong H_2(E(R))$

We have two homotopy fiber sequences:

$$\begin{array}{ccccc} BE(R) & \longrightarrow & B\mathrm{Aut}(S) & \longrightarrow & P_1\, BS^{-1}S_0 \\ \downarrow & & \downarrow & & \| \\ F & \longrightarrow & B\mathrm{Aut}(S)^+ & \longrightarrow & P_1\, BS^{-1}S_0 \end{array}$$

- $BS^{-1}S_0$ denotes basepoint component
- P_1 denotes first Postnikov truncation
- $E(R) = [\mathrm{Aut}(S), \mathrm{Aut}(S)]$
- F is defined to be homotopy fiber on the bottom row
- Vertical maps are homology isomorphisms
- $\pi_1(F) = 0$; $\pi_2(F) \cong \pi_2(B\mathrm{Aut}(S)^+)$

$$H_2(E(R)) \cong H_2(BE(R)) \cong H_2(F) \cong \pi_2(F) \cong \pi_2(B\mathrm{Aut}(S)^+) = K_2(S)$$

Recap of background

- Algebraic K groups of a ring connect geometric and number-theoretic information.
  - $K_0(R)$ is group-completion of f.g. projective modules.
  - $K_1(R)$ is abelianization of $GL(R)$.
  - $K_2(R)$ is H_2 of elementary matrices (commutator).
  - $K_n(R) = \pi_n\left(BGL(R)^+\right)$.
- Quillen's "$+ = S^{-1}S = \mathrm{Q}$"
  - Different but equivalent defintions of higher K groups give different calculational and conceptual information.
  - Bridge, $S^{-1}S$, is an algebraic construction whose classifying space realizes both $+$ and Q constructions.
  - Both homotopical and algebraic tools inform computation of $K_*(R)$.

Motivations for higher-categorical algebra

Who could ask for anything more?!

- Relative K-theory: Given a map of rings $f : R \to T$,

$$K_n(f) = \pi_n\big(\mathrm{hofib}(BGL(R)^+ \to BGL(T)^+)\big)$$

- Hermitian K-theory: If R is a ring with involution (e.g. complex conjugation), Hermitian K-theory is defined via homotopy fixed points of $BGL(R)^+$.
  - Relates to topology of manifolds (e.g. diffeomorphism classes)
  - Relates to motivic homotopy theory
- Use Postnikov P_2 to make an algebraic calculation of $K_3(R)$? (Recall $K_3(\mathbb{Z}) = \mathbb{Z}/48 \twoheadrightarrow \pi_3^s = \mathbb{Z}/24$.)

Are there categories whose classifying spaces compute these?!

Symmetric monoidal algebra in dimension 2

Symmetric monoidal 2-categories are like symmetric monoidal categories, only more so.
Product on objects, morphisms, and 2-morphisms...

Example *Bimod* has objects which are rings; $\mathit{Bimod}(R, T)$ is the category of (R, T)-bimodules.

- Tensor product provides "composition" of $R \xrightarrow{M} T \xrightarrow{N} V$.
- Tensor product of rings provides symmetric monoidal structure.

Example *Cat* has objects which are categories; $\mathit{Cat}(C, D)$ is the category of functors and natural transformations.

- Cartesian product of categories provides a symmetric monoidal structure.

Symmetric monoidal algebra in dimension 2

Better examples:

- relative constructions
- fixed point constructions
- telescope (colimit) constructions

$S^{-1}S$ for 2-categories

Let S be a symmetric monoidal 2-category and suppose:

- all morphisms and 2-morphsisms are invertible (S is a 2-groupoid)
- S has faithful translations

Theorem (Gurski-J.-Osorno)

There is a symmetric monoidal 2-category $S^{-1}S$ with

$$S \longrightarrow S^{-1}S$$

inducing a topological group-completion.

- group-completion on π_0
- isomorphism on localized homology

$S^{-1}S$ for 2-categories

Sketch proof

- $S^{-1}S$: same idea as 1-categorical case, but include 2-morphisms instead of equivalence relation on morphisms.
- Use 2-categorical comma construction to analyze fibers. (This took a while to figure out.)
- Homology spectral sequence collapses after inverting $\pi_0(BS)$. (Just like 1-categorical case.)

$+ = S^{-1}S$ for 2-categories

Given a sequence of objects and faithful functors

$$\cdots \hookrightarrow \mathrm{Aut}_S(A_n) \hookrightarrow \mathrm{Aut}_S(A_{n+1}) \hookrightarrow \cdots,$$

we have a categorical group $\mathrm{Aut}(S) = \mathrm{colim}_n \mathrm{Aut}_S(A_n)$.
(grouplike monoidal groupoid)

If (as in examples), the sequence is cofinal, we have:

Theorem (90% done; Fontes-Gurski-J.)

$$BS^{-1}S \simeq K_0(S) \times B\mathrm{Aut}(S)^{+}$$

Key step Compare colimit of 1-object 2-categories, v.s. 1-object 2-category of colimit (Σ means 1-object "suspension")

$$\mathrm{colim}_n \Sigma \mathrm{Aut}_S(A_n) \quad \text{v.s.} \quad \Sigma \mathrm{colim}_n \mathrm{Aut}_S(A_n)$$

(1-categorical case: groups v.s. 1-object groupoids, is completely straightforward)

Application to K_3

Theorem (In progress; Fontes-J.)

There is a commutator subcategory E such that the following is a homotopy fiber sequence.

$$BE \longrightarrow B\mathrm{Aut}(S) \longrightarrow P_2\, BS^{-1}S_0$$

Corollary

$$K_3(R) \cong H_3(BE)$$

Conjecture

The commutator category E is a categorification of the Steinberg group.

- $St(R)$ is the universal central extension of the commutator subgroup of $GL(R)$.
- Gersten (1973) proved $K_3(R) \cong H_3(St(R))$ using very different methods.

Conclusion

Algebraic K-theory for 2-categories
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- Background on algebraic K-theory
  - K_0, K_1, K_2
  - $BGL(R)^+$
  - $+ = S^{-1}S = \mathbb{Q}$
- Sketch of 2-categorical K-theory
  - Motivations: K_3, Hermitian, relative
  - Current issues: colimits of monoidal 1-categories
  - Future plans: understanding the Steinberg group

Thank You!