Complex numbers

Lang chapter 15

Recall the complex numbers $\mathbb{C} = \mathbb{R}(\sqrt{-1})$.

Basic arithmetic

- real part, imaginary part
- addition, subtraction (like vectors)
- multiplication (foil)
- division (rationalize denominator)
- absolute value
- argument (angle)

Euler's formula

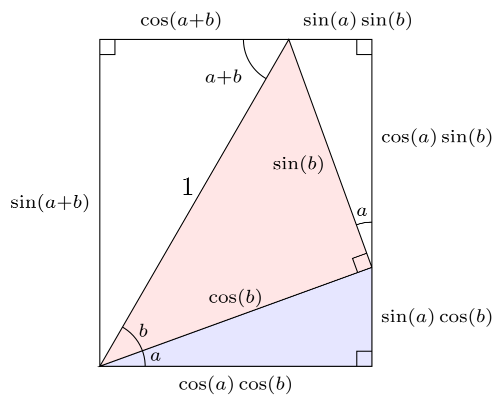
$$e^{it} = \cos(t) + i\sin(t).$$

- Explanation using Taylor series:

$$\begin{aligned} e^x &= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \tfrac{1}{5!}x^5 + \tfrac{1}{6!}x^6 + \cdots \\ \sin(x) &= 0 + x + 0x^2 - \tfrac{1}{3!}x^3 + 0x^4 + \tfrac{1}{5!}x^5 + 0x^6 + \cdots \\ \cos(x) &= 1 + 0x - \tfrac{1}{2}x^2 + 0x^3 + \tfrac{1}{4!}x^4 + 0x^5 - \tfrac{1}{6!}x^6 + \cdots \end{aligned}$$

- Explanation using angle-sum formulas:

$$\begin{aligned} \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \end{aligned}$$



Geometry of complex arithmetic

- addition and subtraction: translation
  For real numbers a, b, c, d, we have

$$(a+ib)+(c+id)=(a+c)+i(b+d).$$

- multiplication and division: scaling and rotation
  For real numbers p, t, q, s, we have

$$pe^{it}\cdot qe^{is}=pqe^{i(t+s)}.$$

Exercise 1 *Observe that multiplication by a complex number re^{it} is a linear function (because of the distributive property)! Therefore, if we view complex numbers $a+ib$ as vectors $\binom{a}{b}$, then we can write multiplication by re^{it} as a matrix. Do this for numbers like $-e^{\pi/4}$ and $3e^{i2\pi/3}$.*