# Functions

*Some of this is covered (better) in the first two sections of Lang, chapter 13. But the parts about binomial coefficients and rational functions aren't there.*

## Definition of function

A function has a *domain* and a *range*. A function from domain $D$ to range $R$ is an association which assigns, to each member of $D$, a member of $R$. If we have a function named $f$, we write $f(x)$ (say "$f$ of $x$") for the element of $R$ that is assigned to $x$. We write

$$f \colon D \to R$$

to indicate that $f$ is a function from $D$ to $R$. We also say $x$ is the *input* and $f(x)$ is the *value* of $f$ at $x$ or the *output* of $f$ at $x$. We write

$$x \mapsto f(x)$$

to indicate that the function $f$ assigns the element $x \in D$ to the element $f(x) \in R$.

Most of the functions we encounter will be functions from $\mathbb{R}$ to $\mathbb{R}$, but sometimes our functions will be defined only for subsets of $\mathbb{R}$, like the subset of *positive* numbers, or the subset of *nonzero* numbers. Later, we might encounter some functions whose domains and ranges are things other than numbers, but numbers are a good place to start.

### Examples

The function which assigns each number $x$ to its square, $x^2$, is the function

$$x \mapsto x^2.$$

The function which assigns each number $x$ to the number 3 is a constant function

$$x \mapsto 3.$$

The function which assigns to each number $t$ the line through $(0, 1)$ with slope $t$ is a function from $\mathbb{R}$ to the set of all lines through $(0, 1)$.

**Remark 1.** *If, as in the first two examples, we can express the values of a function in terms of some arithmetic on the inputs, then we might simply write something like "Consider the function $g(x) = x^2 - 1$" to mean that $g$ is the funciton which assigns $x^2 - 1$ to the input $x$. It's easy to confuse the outputs, $g(x)$, with the function itself, $g$. One way to keep them separate is to think of $g$ as a machine, and $g(x)$ are the things the machine produces.*

# Polynomial functions

A polynomial function is a function of the form

$$x \mapsto a_n x^n + \cdots + a_1 x + a_0$$

for some natural number $n \geq 0$ and some constants $a_0, \ldots, a_n$. The numbers $a_i$ are called the *coefficients* and the number $n$ is called the *degree*.

**Example 1.**

- $g(x) = x^2 + 1$ *is a polynomial of degree 2, with coefficients* $a_2 = 1$, $a_1 = 0$, *and* $a_0 = 1$.
- $p(x) = 3x^3 - 2x - 4$ *is a polynomial of degree* 3 *with coefficients* 3, 0, $-2$, *and* $-4$.
- *The constant funciton* $k(x) = 17$ *is a polynomial of degree 0.*
- *The function* $f(x) = (x - 3)^2$ *is a polynomial, even though we didn't write it in polynomial form. It has degree 2; and its coefficients are* 1, $-6$, *and* 9.
- *The function* $r(x) = \sqrt{x}$ *is* not *a polynomial. This means that it* can't *be written in the polynomial form above, and it's something that requires explanation. If you know about derivatives, it's easy to use them to prove that* $\sqrt{x}$ *isn't a polynomial...*

## Making new polynomials

Here's an interesting and useful fact about polynomials: The sum or product of any two polynomials is another polynomial. Why is this? The distributive property! Likewise, a constant multiple of a polynomial is a polynomial.

Note, however, that the quotient of two polynomials is not necessarily a polynomial. For example, the function

$$f(x) = \frac{1}{x^2 + 1}$$

is *not* a polynomial. Again, you can see this easily by considering its derivatives, if you know about those.

Here's an interesting consequence of these observations: we know the function $(x+1)^5$ is a polynomial, even if we don't know what its coefficients are. Likewise, $(x^2 - 3)^6 (x^3 + x^2 + x + 1)^8$ must be a polynomial.

# Interlude: the binomial theorem

The polynomials $(x+a)^n$ are special and important, and there's actually a formula for their coefficients. They are called the *binomial coefficients* because they come from a power of a two-term sum. Before we state the theorem, let's make a few observations:

- $(x+a)^0 = 1$, and $(x+a)^1 = x + a$
- $(x+a)^2 = x^2 + 2ax + a^2$
- $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$
- $(x+a)^n = x^n + \cdots + a^n$.
- all the middle terms of $(x+a)^n$ are of the form $B \cdot a^{n-k}x^k$ for some number $B$ that depends on $n$ and $k$.
- $x^n = a^0x^n$ and $a^n = a^nx^0$.

The binomial theorem gives a formula for all these numbers $B$, and it's pretty cool.

**Theorem 1.** *The coefficient of $x^k a^{n-k}$ in the polynomial expression for $(x+a)^n$ is given by the binomial coefficient*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The expression here is very exciting because it uses the *factorial* symbol! The factorial is a function written with an exclamation point, and it is defined by

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

For example,

- $1! = 1$
- $2! = 2$
- $3! = 3 \cdot 2 \cdot 1 = 6$
- $4! = 4 \cdot 3! = 24$
- $5! = 5 \cdot 4! = 120$
- $6! = 6 \cdot 5! = 720$

Now here's a special note: 0! is defined to be 1, not 0. There are so many good reasons for this, but they will digress us, so for now I'll just say that this convention makes the formula for $\binom{n}{n}$ and $\binom{n}{0}$ correct (both are 1).

**Exercise 1** *Use the binomial formula to compute* $\binom{4}{k}$ *for* $k = 0, 1, 2, 3, 4$, *and check that these are the correct coefficients for* $(x + a)^4$.

## Choices and Pascal's triangle

We have seen (or will see) these binomial coefficients in another context: when we learn about probability, we learn that we need to count all the ways of choosing $k$ things from among $n$ – for example, all the ways of choosing a $k$-item lunch out of a bag that contains $n$ food items. It turns out that the number of different lunches which can be made this way is precisely the number $\binom{n}{k}$. For example, there are 6 ways to choose 2 items from among 4, and indeed $\binom{4}{2} = 6$.

These numbers are sometimes written in a triangle, called *Pascal's triangle*, and they have an amazing pattern!

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & \cdots & & & & & & & & & \\
1 & & 8 & & 28 & & \cdots & & & & & & & & & &
\end{array}
$$

Each number is the sum of the two numbers above it!!

**Exercise 2** *Use the formula for binomial coefficients to prove that they fit the pattern in Pascal's triangle. That is, prove that the expressions*

$$\binom{n}{k} + \binom{n}{k+1}$$

*and*

$$\binom{n+1}{k+1}$$

*are equal for all natural numbers $n$ and $k$ with $0 \leq k \leq n$.*
*Hint: Try some specific numbers for $n$ and $k$ if you want to get a feel for how this works in general – it just requires a little algebra to add fractions.*

### Proving the binomial theorem

The binomial theorem says that the coefficients of $(x+1)^n$ are given by the numbers $\binom{n}{k}$. To explain why this is, we can use either one of two ideas.

- Idea 1: Explain why the coefficients of $(x+a)^n$ have the same pattern as in Pascal's triangle, by considering $(x+a)^{n+1} = (x+a)^n \cdot (x+a)$.
- Idea 2: Explain why the coefficient of $a^{n-k}x^k$ in $(x+a)^n$ must be the number of ways of choosing $k$ things from among $n$, and then in some other class explain why this number of choices is given by $\binom{n}{k}$.

Both of these are good ideas, and I think we'll discuss both of them in class.

## Roots and factors of polynomials

Returning to general polynomials, there are a few more things to say and they all have the following theme: *the arithmetic of polynomials is very very much like the arithmetic of whole numbers.*

- Theorem: If $f$ is a polynomial and has a root at $x = c$, then $f$ factors as $f(x) = (x-c) * g(x)$ for some polynomial $g$ with degree smaller than degree of $f$
- Corollary: If $f$ is the zero polynomial, then all the coefficients of $f$ are zero.
- Corollary: If $f$ has two different expressions as a polynomial, then all the coefficients of the two expressions must be equal.
- Compare: digits of a number v.s. fractions
  Two base-ten whole numbers are equal if and only if they have the same digits. On the other hand, fractions can be equal (equivalent) even though

their numerators and denominators are different. Polynomials are more like whole numbers than fractions. In fact, polynomials are like numbers written in *base* $x$!!! (That's a cool idea.) If you wrote algorithms for the coefficients of adding, subtracting, and multiplying polynomials, they would be the algorithms for place value arithmetic!

- Euclidean algorithm (polynomial long division): This is the thing that allows us to tell when a quotient of two polynomials is a polynomial! It is very very much like long division.

# Rational functions

A quotient of polynomials is called a *rational function*, for the same reason that a quotient of whole numbers is called a *rational number*. As we know, sometimes a fraction can actually be simplified to a whole number, and sometimes not.

**Exercise 3** *Use polynomial long division to decide whether or not the following quotients are actually polynomials, or not.*

- $\frac{x^3+3x^2+3x+1}{x+1}$
- $\frac{x^4+4x^2+1}{x+1}$
- $\frac{x^4-x^2-2x-1}{x^2+x+1}$

Now we come to another cool connection with elementary arithmetic, geometry, and calculus. As you know, one of the most exciting uses of long division (for numbers) is to find the decimal expansion for fractions such as $\frac{1}{3}$ or $\frac{1}{9}$. The same is true for polynomials!! To do this, we should first understand what the "decimal places" are for polynomials: They are simply the *negative* integer powers, $x^{-1}$, $x^{-2}$, etc.

**Exercise 4** *Use long division to show that the rational function* $\frac{1}{x-1}$ *is equal to the series* $x^{-1}+x^{-2}+x^{-3}+\cdots$. *Verify that, when you substitue* $x=10$, *you get the correct expression for* $\frac{1}{9}$. *When you substitute* $x=4$, *you get the correct expression for* $\frac{1}{3}$. *In fact, when you substitute any number* $x>1$, *you get the correct expression!!*

You can rearrange the identity

$$\frac{1}{x-1} = x^{-1} + x^{-2} + x^{-3} + \cdots$$

into a more familiar form by making the replacement $r = \frac{1}{x}$. Then the right hand side becomes $r + r^2 + r^3 + \cdots$, and is defined for $|r| < 1$. The left hand side becomes $1/(\frac{1}{r} - 1) = \frac{r}{1-r}$, and we have

$$\frac{r}{1-r} = r + r^2 + r^3 + \cdots$$
$$\frac{1}{1-r} = 1 + r + r^2 + \cdots$$

for $|r| < 1$. We've seen this before, when we discussed similarity in geometry, scaling by a factor of $r$. Wow! If you've seen Taylor series in calculus, this is the Taylor series for $\frac{1}{1-r}$. Double wow!!

## Scaling and the geometric series

Imagine you start with a square of side length 1, and scale it by a factor of $r$, for some $r < 1$. Then do this again and again, stacking the squares next to eachother so you get something like this:

If you have infinitely many squares, then their total length is $1+r+r^2+r^3+\cdots$. But also the whole picture can be scaled by a factor of $r$, and, *if you have infinitely many squares*, then you'll get all the squares except the first one. So whatever the total length, $L$, might be, it must solve the following equation: $1 + rL = L$. Therefore we have

$$L = 1 + r + r^2 + \cdots$$
$$L = \frac{1}{1-r}$$

and this recovers the formula for the geometric series above.

The equivalent formulation $\frac{r}{1-r} = r + r^2 + r^3 + \cdots$ is useful for explaining why $.\overline{9} = 1$ (take $r = \frac{1}{10}$ and multiply by 9) or Zeno's paradox $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$ (take $r = \frac{1}{2}$).

## Taylor series

If you haven't yet seen Taylor series, let's leave it at this: Series are another class of functions that includes polynomials, but also sums with infinitely many terms, as above. Lots of interesting functions that aren't polynomials are in this larger class. It includes all rational functions, square roots, trigonometric functions, exponential functions, and logarithms. We have seen how to use long division to express rational functions as series, and in calculus you will learn about the rest of these.

One has to be a little more careful with series than with polynomials, because they generally aren't defined for all values of the input variable, but expressing functions as series is really a powerful way to understand functions.