Hyperbolas

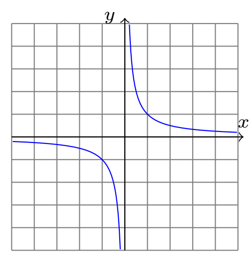
We discuss two different equations for hyperbolas, and how they are related by rotation. This gives us some practice using matrices, algebra, and a little more experience with conic sections. This content appears in Lang §§12.4 and 12.5.

Introduction

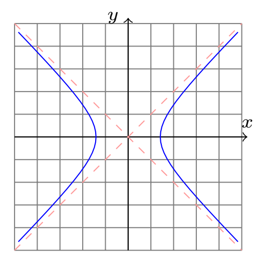
There are two different equations that describe hyperbolas. The first is of the form

$$xy = k$$

for some constant k. The classic $xy = 1$ is shown below, and this can be expressed equivalently as our old friend $y = \frac{1}{x}$, which lead us to discover the logarithm and exponential functions.



$xy = 1$



$x^2 - y^2 = 2$

The second way of describing hyperbolas is with an equation like

$$x^2 - y^2 = a^2$$

for some constant a. This number a is called the semi-major axis, and it's sort of like the "radius" of an "imaginary ellipse". To see what I mean, recall that the equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the numbers a and b are the semi-major and semi-minor axes. The ellipse crosses the x and y axes at $(\pm a, 0)$ and $(0, \pm b)$. When $a = b$, the ellipse is a circle! A hyperbola is the same kind of equation, but with $b = ia$, an imaginary number.

This way of presenting hyperbolas shows that they have all kinds of interesting geometry, which is called *hyperbolic trigonometry*, because the geometry of the circle is called *trigonometry*.

Rotation by $\pi/4$

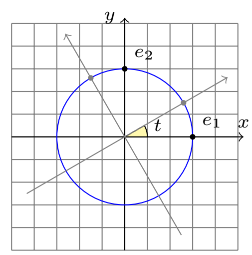
Now the main question I'd like to answer is this: *How do we know that these two equations describe the same kind of shape?* The answer, simply put, is that one is a rotation by 45° of the other. But how do we know *that*? To explain, we will need to understand rotation a little more.

Rotation by any angle preserves vector sum, so it is a *linear transformation.* That means it can be described by a matrix. We will use this to analyze the two equations for hyperbolas.

Suppose we want to rotate by an angle t. To express this as a matrix, we need to know where the two basis vectors

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

go after rotation.



The circle $x^2 + y^2 = 1$ will help us understand rotation by t.

Recalling our experience with polar coordinates, we realize that rotation by and angle t sends e_1 to $(\cos(t), \sin(t))$ and sends e_2 to $(-\sin(t), \cos(t))$. Therefore the matrix for rotation by t is

$$R_t = \begin{pmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{pmatrix}.$$

Exercise 1 *Show that, by the Pythagorean Theorem, this is a matrix of determinant 1*

Examples

Rotations by $\frac{\pi}{2} = 90°$ and $\pi = 180°$ are given by

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

Exercise 2 *Use triangle geometry to understand the matrix for rotation by $45°$.*

Rotating the hyperbola

In the previous exercise, we find that rotation by $\frac{pi}{4}$ (i.e., $45°$) is given by the matrix

$$R_{\pi/4} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Now to understand how this matrix transforms a hyperbola such as

$$x^2 - y^2 = 2,$$

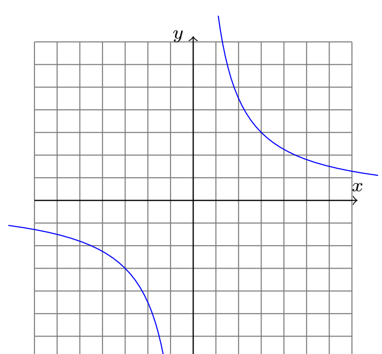
we have two steps.

Exercise 3

(a) *Write a general expression for what happens when we apply rotation by $\frac{\pi}{4}$ to a point (x, y).*

(b) *Suppose that the point (x, y) was a point on the hyperbola; use that to write an equation for the new points $(u, v) = R_{\pi/4}(x, y)$.*
Hint: See if uv simplifies to a pleasant expression

Exercise 4 *Consider the hyperbola $xy = 9$. Rotating this by $\frac{\pi}{4}$ gives another hyperbola. What is its equation? If you like, sketch a picture on the axes below!*



$xy = 9$