Linear equations

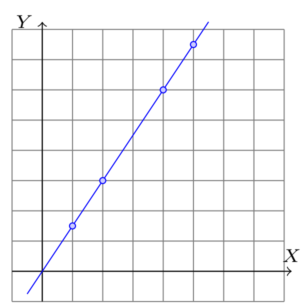
Proportions are special kinds of linear equations

Remember when we learned about proportions? Aren't they just great? Two quantities are *proportional* when they're related by a constant scale factor. We say Y and X are proportional in a ratio of $b : a$ when we have b units of Y for every a units of X. For example, a $3 : 2$ ratio of red paint to blue paint makes a reddish purple. It means that we have 3 units of red for every 2 units of blue.

One of the neat things we started to see when we studied ratios is that you can express, by drawing a graph, all of the quantities that are related by the same given ratio. For example, when Y and X are related by a $3 : 2$ ratio, here are just a few of the quantities we could have.

Y	3	6	$\frac{3}{2}$	$\frac{15}{2}$
X	2	4	1	5

When we plot these on a grid, we see a beautiful shape emerge.



It's a line!

In this picture, we've put dots on the four points listed in the table above, but really every point on the line corresponds to a possible (X, Y) value with the same Y-to-X ratio of 3:2. This includes points with negative coordinates, such as $(-2, -3)$. For an application such as our paint-mixing story, negative quantities might not make sense, but in other applications like distance in a particular direction, or temperature, etcetera, negative quantities could definitely make sense. Mathematically, they are no problem, so we generally include them.

You might have noticed that we are focusing on the Y-to-X ratio instead of X-to-Y. That's why we listed Y before X in the introductory explanation and in the table. We did this because the rate of the Y-to-X ratio is the one that gives us the *slope* of the line (rise-over-run).

You can remember that slope is the Y-to-X rate, and not the X-to-Y one, because both the slope and the Y-to-X rate are saying how much Y for every *one* unit of X. In our example, that's $\frac{3}{2}$. This is the number that allows us to quickly figure out the Y amount for any given X amount. The Y amount is always $\frac{3}{2}$ times the X amount; this can be written succinctly as

$$Y = \tfrac{3}{2}X.$$

If you wanted to get the X amount for a given Y amount, you could of course rearrange this equation, and then you would be multiplying Y by $\frac{2}{3}$—the rate for the X-to-Y ratio. We'll just focus on the first one because people usually like to graph functions with the inputs on the horizontal axis and the outputs on the vertical axis.

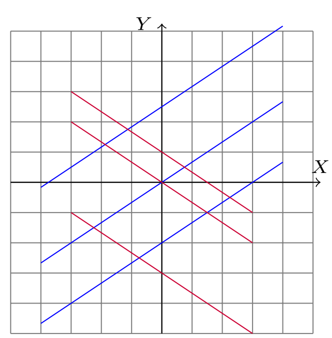
The equation

$$Y = \tfrac{3}{2}X$$

is called *linear* because it's graph is a straight line. Every proportion gives a linear equation in this way, and the graph is always a straight line passing through $(0, 0)$.

Linear equations are linear

There are more straight line graphs than just the ones passing through $(0, 0)$, but not very many. Here are several in a picture.



When we say there are "not very many more", maybe that's a little misleading. Indeed, there are infinitely many more, and for each point b on the Y-axis, there are just as many lines passing through $(0, b)$ as there are lines passing through $(0, 0)$. But what we mean is that, once you've understood all the lines passing through $(0, 0)$ (they are determined by their slope), then it's not very much harder to understand all the lines passing through $(0, b)$—they're just shifted by b (up if b is positive or down if b is negative).

This is why every possible line is given by the graph of some equation of the form

$$Y = mX + b$$

for some numbers m and b. The coefficient m is what determines the slope of the line, and the constant term b is what determines the Y-intercept.

Exercise 1 *Graph the following lines*

(a) $Y = \frac{3}{4}X - 2$

(b) $X + Y = 1$

(c) $X + 2Y + 1 = \frac{Y}{3}$

Exercise 2 *Write equations for the following lines*

(a) *The line passing through* $(0, 1)$ *and having slope -1.*

(b) *The line passing through* $(2, 2)$ *and having slope 1.*

(c) *The line passing through the points* $(-1, 0)$ *and* $(2, \frac{5}{2})$.

As you can see in these exercises, linear equations don't always present themselves in the slope-intercept form we described above. They can always be rearranged to that form, and that may or may not be a good idea depending on what you're trying to do.

This leads to an alternative way of characterizing linear equations: an equation is linear (has a straight line graph) if it involves two variables that are multiplied by constants and added to eachother. So $a - 6b = 4$ is linear, and so is $u + v = 0$, but $u + uv = 0$ is not linear, nor is $u^2 - 2u = 0$.

We also use the term "linear" for equations involving more variables, but still just multiplied by constants and added. For example $z = 3x + 2y + 1$ is called linear. It's graph is a plane, which is really sort of a 2D line anyway. It's fun to think about something like $w = x + y + z - 2$ and what that graph would be. Whatever it is, it's "flat" in the same way that lines and planes are.

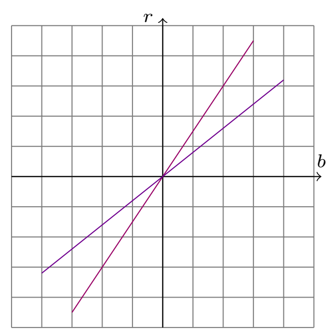
Simultaneous equations

Up until now, when we've discussed proportions we've just discussed one ratio at a time. So we might talk about one paint mixture in a $3:2$ ratio, which gives the equation (firstquantity) $= \frac{3}{2}$(secondquantity), and another paint mixture in a $4:5$ ratio, which gives the equation (firstquantity) $= \frac{4}{5}$(secondquantity). But there are interesting situations where we want to talk about quantities satisfying *two equations at the same time*! This is called solving *simultaneous equations.*

Actually, for ratios, it's not very interesting to have an amount satisfying two different ratios at the same time.

Exercise 3 *Suppose you want to make a red/blue paint mix which is in* both *a $3:2$ ratio and a $4:5$ ratio. Use arithmetic and/or proportional reasoning to explain that the only way to do this is with the (blue, red) quantities being* $(0,0)$.

The reason that $(0,0)$ is the only solution is easy to see on a graph. If we let r be the red amount and b be the blue amount, then the $3:2$ ratio means amounts that are on the line $r = \frac{3}{2}b$. Meanwhile, the $4:5$ ratio means amounts that are on the line $r = \frac{4}{5}b$. Here's a graph of these two lines.



A (blue,red) amount satisfying both ratios would have to be one that corresponds to a point on *both* lines. And the graph shows us that there is just one such point, at (0,0).

Although this application has an uninteresting answer, there are lots of other applications where solving simultaneous linear equations gives really interesting answers.

Exercise 4 *A dog sees a cat which is 20 feet away and starts running toward it. The dog runs at a speed of 5 feet per second, and the cat, which starts running at the same time, runs at a speed of 3 feet per second. (Assume the dog is running straight toward the cat, which is running straight toward a tree, so that we don't have to think about curved paths yet.)*

The dog wants to give the cat a hug. If the cat starts 10 feet away from the tree, does it reach the tree before the dog hugs it?

Explain what happens by writing equations and drawing their graphs. There are lots of ways you might do this, but one good one is to call the dog's starting point A, and write an equation for his distance from A as a function of time. Then write another equation for the cat's distance from A as a function of time. Because both equations involve the same quantities (time and "distance from A") it is meaningful to graph them on the same set of axes.

What does the event "dog hugs cat" correspond to on your graph? How does the cat's initial distance from the tree present itself on your graph? If you were going to rewrite this problem with a different starting distance from the dog to the cat or from the cat to the tree, what would be some exciting distances to choose?

Exercise 5 *Graph the following equations on the same axes and discuss their simultaneous solutions.*

(a) $x + y = 2$ *and* $x - y = 4$

(b) $y = \frac{-1}{3}x + 1$ *and* $y = \frac{-1}{3}x - 1$

(c) $y = 3x + 1$ *and* $3(x + y) = 4y - 1$

(d) $x + y = 2$ *and* $x - y = 4$ *and* $y = x - 1$

When "solving" equations, you are looking for values—numbers—that make the equations hold. This might seem different from when we graph equations, and the variables represent a whole range of values. In that case, the graph is showing all possible pairs of points that make the equation true. When solving simultaneous equations, we are looking for points that make *all* of the equations true. This means we are looking for all possible points that lie on *all* the lines. These are known as *intersections*. Really this isn't such a different way of thinking about what the variables mean; it's just that in a lot of examples of solving equations, there is only one possible value that makes the equations hold, so "all possible values" means "the only possible value".

If anything, this is a difference of mindset instead of mathematics. When graphing, we might be thinking "here's a left-hand and a right-hand side; let's mark all the points where the two sides are equal". When solving, we might instead be thinking "let's assume we have numbers that satisfy this equation; by rearranging the equation(s) we will learn what values these numbers have to be".

Remark 1. *It might be interesting to think about how "solving" an equation in one variable, like $x + 5 = 2(x - 1)$ is really the same as "graphing" it on the x-axis. The point $x = 7$ is a 0-dimensional line.*

In interesting cases there is just one point that lies on all the graphs we're considering. But sometimes there are none, and sometimes there are multiple ones (maybe even infinitely many). For two linear equations, there are only three possiblities:

- lines can have one intersection point (one solution), or
- the lines can be parallel (no solution), or
- the lines are exactly the same (infinitely many solutions).

If you have more variables and/or more equations, you can still ask whether they have simultaneous solutions, and you can still ask what that means about their graphs.

Exercise 6 *Consider the equations $x + y + z = 0$ and $3x + 2y + z = 0$ and $x - y - z = 0$. Describe the simultaneous solutions to the first two, and the simultaneous solutions to all three.*

Strategies for solving simultaneous equations

When solving simultaneous equations, there are two basic strategies (procedures) that you could use. Both are consequences of the following basic facts:

- The equal sign means equal.
- Simultaneous solutions are simultaneous.

Substitution

One strategy for solving simultaneous equations is called "substitution" or "elimination". Basically you rearrange one equation to have one variable expressed in terms of the others, and then substitute that expression for that variable in the other equations. This gives you one fewer equation, and one fewer variable. Repeat until you have one equation in one variable, and then solve it. After that, work back through your previous equations, using the values you've determined previously, to solve for each of the other variables.

Arithmetic with equations

Another strategy, which I always think seems cooler but is really the same, is to add, subtract, and multiply whole equations. What this means is to do some arithmetic with one side of two equations, and then do that same arithmetic to the other side of the same equations. Because you've done the same arithmetic to quantities which were equal, you get new quantities which are still equal.

This method can seem more powerful because it can eliminate variables without having to go through the same steps as substitution. But it can also be unwieldy because you quickly produce lots and lots of equations and it can be tricky to tell which ones are redundant.

Exercise 7 *Solve the following sets of simultaneous equations.*

(a) $2s + 4t = 5$ *and* $s - t = 2$

(b) $u + v - w = 1$ *and* $u + v = w - v$ *and* $2u - 3v = w$