Matrices

Lang chapter 17.

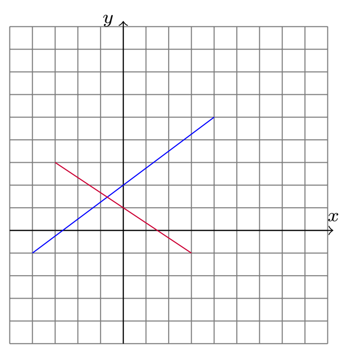
The geometry of systems of linear equations.

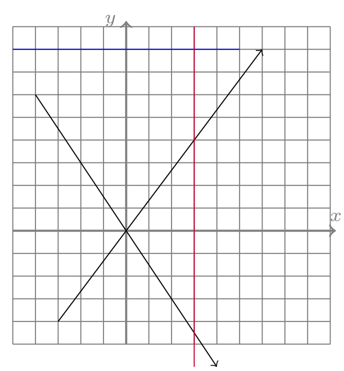
- Recall systems of linear equations and direct method of solution
- Recall some systems have no solutions, some have many, but generally a system has exactly one solution

When we discussed systems of linear equations, we observed that each equation defines a line, and a solution to the system is the intersection of the lines. For example the system

$$3y + 2x = 3$$
$$4y - 3x = 8$$

can be visualized as the intersection of the two lines below (left).





Another, more abstract way to think about this is to define a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ as $f(x, y) = (3y + 2x, 4y - 3x)$. This function sends the axes to the lines indicated at right, and the red/blue lines to the indicated horizontal/vertical lines.

Linear transformations

- Observe that a system of expressions defines a function (*transformation*)
- When the system is linear, the transformation is linear

- Plan: analyze linear transformations; develop a systematic solution to *all* systems of linear equations

A function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called *linear* if it has the following properties:

- For any two vectors A and B, $f(A+B) = f(A) + f(B)$.
- For any real number r and vector A, $f(rA) = rf(A)$.

Exercise 1 *Show that these properties have the following consequences:*

- $f(A-B) = f(A) - f(B)$.
- $f(0,0) = (0,0)$.
- $f(x,y) = xf(1,0) + yf(0,1)$.

The two vectors $(1,0)$ and $(0,1)$ are called *basis vectors* because, for any linear function f, the entire function is determined by its value on these two points. In the example of a linear function given by two linear expressions

$$\begin{gathered} ax + by \\ cx + dy \end{gathered}$$

we have $f(x,y) = (ax+by\,,\, cx+dy)$.

Exercise 2 *Check that the function f defined by two linear expressions above is indeed linear.*

Note that $f(1,0) = (a,c)$ and $f(0,1) = (b,d)$. Therefore we can recover the coefficients of the linear expression by the values of f on the two basis vectors.

The *matrix* for a linear funciton f is a grid of numbers, where the numbers are the coordinates of the values of f on basis vectors. In the example above, this is the following:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To compute the value of f on any vector (x,y), we use matrix multiplication with the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

Matrices as linear transformations

- Matrices (grids of numbers)
- Every linear transformation is given by a matrix
- Matrix arithmetic and geometry
- Examples: Dilation, rotation, combinations thereof

Determinants arise from matrices

- Computing determinants
- Determinants determine solutions!
- Inverting matrices: reversing a linear transformation

The determinant of a 2x2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

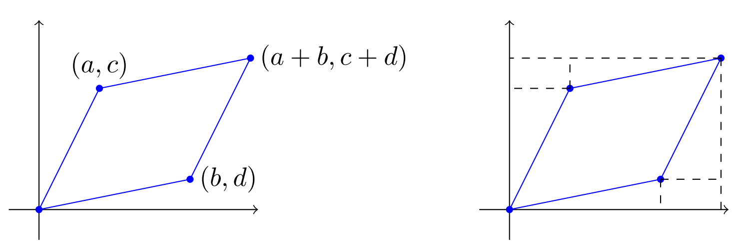
is the number $\det(A) = ad - bc$. This number is important for the following reasons.

Proposition 1 (Determinant in multiplicative)**.** *For any two matrices A and B,*

$$\det(AB) = \det(A)\det(B).$$

Proposition 2 (Determinant measures area)**.** *Let (a, c) and (b, d) be the column vectors of a matrix A. Let P be the parallelogram defined by the origin, these two vectors, and their sum. Then the area of P is given by $|\det(A)| = |ad - bc|$.*

Exercise 3 *Use the following picture and algebra to prove the previous proposition*



Exercise 4 *Consider the possible options for where (a, c) and (b, d) might be in the plane. What orientation(s) of the parallelogram correspond to $\det(A)$ being positive, and what orientation(s) correspond to $\det(A)$ being negative?*

Theorem 1. *The matrix A has an inverse if and only if its determinant $\det(A)$ is nonzero. In this case, the inverse of A is given by*

$$\frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof It's easy to check that, if $\det(A)$ is nonzero, the given formula defines an inverse. The converse—that A is invertible *only* if $\det(A)$ is nonzero—follows from the proposition above. If $\det(A) = 0$, then the parallelogram determined by the columns of A has zero area, so the two columns lie on the same line and A cannot have an inverse. ■

Exercise 5 *Give an alternate proof of the theorem simply using the fact that the determinant is multiplicative.*

And finally, here is how we can use the determinant to determine solutions to systems of linear equations: Given a system like

$$\begin{aligned} ax + by &= u \\ cx + dy &= v, \end{aligned}$$

then we take the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and, if $\det(A) \neq 0$, we define the inverse matrix

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then the system of equations is simply

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and its solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} u \\ v \end{pmatrix}.$$