Segments, Rays, and Lines

We can use the arithmetic of vectors to understand lines better.

Classic geometry (the kind that Euclid did) studies points, lines, and other geometric figures by starting with a list of their properties (axioms) and making logical deductions from those properties. This is known as *synthetic* geometry, and we saw some of it in 1136 (and also probably in high school geometry).

When we introduce coordinates and use algebra to study geometry, we're doing what's known as *analytic* geometry. The vector arithmetic we introduced in the previous section is a part of that. In this section, we'll see how to use our vector arithmetic to make some basic calculations about lines.

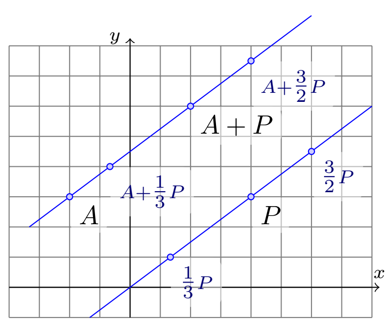
Background

- Recall vector arithmetic
- Recall geometric concepts *point* and *line*

Lines in coordinates

Suppose that $P = (P_x, P_y)$ is a point in the plane. Consider the set of points $t \cdot P$ for $t \in \mathbb{R}$. This set of points is the set of all scalar multiples of P, and together it forms a line passing through P and $(0, 0)$.

Now suppose that $A = (A_x, A_y)$ is another point in the plane. Consider the set of points $A + t \cdot P$. What shape does this form?



It's a line!

Exercise 1 *How could you use a formula like this to make the line passing through the points A and P?*

Rays and Segments

Rays and segments are parts of lines, so to express these we just restrict the possible values of t.

Exercise 2 *What restriction on t in the expression above would produce the ray which begins at A and passes through $A + P$?*

Exercise 3 *Suppose that P and Q are two points in the plane.*

(a) *Observe that the expressions $tQ+(1-t)P$ and $P+t(Q-P)$ are equivalent.*

(b) *Observe that for $t \in [0, 1]$, the points given by these expressions form the line segment between P and Q.*

Finally, how do we connect this way of expressing lines with the equations for lines we've written before? For example, suppose I have a line written in point-slope form, such as the line

$$y = 2x - 1.$$

Since this line passes through the points $(0, -1)$ and $(1, 1)$, it is the same as the line

$$t \cdot (1, 1) - (1 - t) \cdot (0, 1)$$

which is equal to

$$(t, t) - (0, 1 - t)$$

or

$$(t, -1 + 2t).$$

In other words, the line $y = 2x - 1$ is the set of points $(t, 2t - 1)$! This is exactly what we already knew, just writing the variable x as t instead. Observe, moreover, that the arithmetic we've done shows how to find the equation for the line passing through any two points.

Exercise 4 *Consider the line given by $t \cdot (1, 1) + (1 - t) \cdot (-2, -1)$ for $t \in [0, 1]$. Use vector arithmetic to find the equation for this line.*

This section has two purposes which I hope are clear by now:

- Use vector arithmetic to easily express the points on the line between any two given points.
- This vector arithmetic is how we find the point-slope form of the equation for a line.

Applications

Now we're ready for some applications.

- Find where line crosses axes
- Find where two lines cross
- Find where line crosses circle