The quadratic formula

Polynomials

Expressions that can be formed using addition and multiplication of a variable are called *polynomials*. For example, $3 + x^2$ is a polynomial with the variable x. A polynomial with the variable t might be $2t + t^3 - 1$. In general, a polynomial has the form

$$a_n x^n + \cdots + a_1 x + a_0$$

where each of the a_i is a number, x is the variable, and each exponent of x is a natural number. Things like $2(x-1)^2 + x$ don't look like this, but when expanded out they have this form. On the other hand, expressions like $x^2 - \frac{1}{x}$ or $\sqrt{x} + 2$ are not polynomials because no matter how you rearrange them they won't be expressable in the form above.

The largest exponent of a polynomial is called its *degree*, and we've already studied degree 1 polynomials: those are the linear expressions. Degree 2 is called *quadratic*, and that's what we'll focus on now. The main ideas are the following:

- Examples where quadratic polynomials arise
- The graph of a quadratic polynomial is a parabola
- Two ways of understanding quadratic polynomials: their *coefficients* and their *roots*
- The quadratic formula is a way of expressing roots in terms of coefficients

Motivation for quadratic polynomials

Quadratic polynomials

- parabolas: e.g., points equidistant from $(0, 1)$ and x-axis
- acceleration due to gravity: parabolic motion
- areas: rectangular frames and the golden ratio
- intersection of lines and quadratic curves: e.g. the circle of radius 3 and the line $y = 2x + 5$.

Roots

Understanding the values of a quadratic function tells us how to answer questions about things like the examples above. Notice that trying to solve an equation like

$$(x-1)^2 + 2 = 4$$

Is really just the same as solving

$$(x-1)^2 - 2 = 0.$$

So finding a particular value of some quadratic function is really just the same as finding the zeroes of some slightly different quadratic function. That's why we focus so much on zeroes—it's a special case that solves the general problem. And that's why zeroes have a special name—the *roots*.

A parabola can cross the x-axis twice, just once, or not at all, and—if we're restricting ourselves to real numbers—a quadratic polynomial can have two roots, just one (repeated twice), or none at all.

Exercise 1 *Sketch graphs of the following quadratic functions*

- $(x-1)(x+1)$
- $(x-2)(x+3)$
- $(x+1)^2$
- $(x+1)^2 + 1$

The quadratic formula

The roots of a quadratic polynomial

$$ax^2 + bx + c$$

are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

I would like to explain where this fomula comes from, because it is a simple idea and I think it helps give some meaning to the formula.

Completing the square

The simple idea has two parts: First, is that it is easy to solve equations where there is no linear term, such as

$$x^2 - 4 = 0 \quad \text{or} \quad (x-1)^2 - 4 = 0.$$

Second, every quadratic polynomial can be rearranged into this form. This part takes a little more explaining. For example, consider something like $x^2 - 2x + 1$. You can tell this is equivalent to $(x-1)^2$ because squares like this always have the form $x^2 + 2kx + k^2$. But if you had just $x^2 - 2x$, that's not a square, but it is equal to $x^2 - 2x + 1 - 1$, and that means it's equal to $(x-1)^2 - 1$. Think about how we knew to add and subtract 1. We chose 1 because it's the number that completes the square.

Exercise 2 *Complete the square in the following examples.*

- $x^2 - 2x + 3$
- $x^2 + 6x - 1$
- $x^2 + 3x - 1$
- $x^2 + rx + s$ *for constants* r *and* s*.*

General case

For the general quadratic formula, notice that

$$ax^2 + bx + c = a(x^2 + \tfrac{b}{a}x + \tfrac{c}{a}).$$

So the roots of this polynomial are the same as the roots of

$$x^2 + \tfrac{b}{a}x + \tfrac{c}{a},$$

and we can easily complete the square and solve for these roots. This gives the quadratic fomula.

Further remarks

The expression $b^2 - 4ac$ is called the *discriminant*, because it discriminates how many real roots a quadratic polynomial has.

When people were studying polynomials, they noticed that sometimes, even when the discriminant is negative, you could still do arithmetic with those "imaginary" roots and answer real questions about the polynomial. This was before people knew about the complex numbers, and sometimes they would actually hide their work with negative discriminants, so that other mathematicians wouldn't shame them. The observed that these "imaginary" roots behaved in ways that were sort of like regular "real" numbers, and sort of not. It took quite a while to realize that a number system like $\mathbb{Q}(\sqrt{-1})$ could be just as "real" as one like $\mathbb{Q}(\sqrt{5})$.

There are some formulas like the quadratic formula for higher-degree polynomials, but, importantly, *only for degrees 3 and 4*! The discovery that there *cannot* be a formula for higher degree polynomials was an important milestone, partly because the question itself is so important, but also because the new idea that explained why not was so profoundly different from the ways that people had done mathematics previously. This idea, which is now named after its inventor, Everiste Galois, is a cornerstone of modern mathematics.

Historical notes on cubic, quartic, quintic

Polynomials of degree 3, 4, and 5 are called *cubic*, *quartic*, and *quintic*, respectively. There is a fascinating history of people trying to understnd these polynomials, particularly their roots. It includes fame, fortune, betrayal, genius, and a tragic ending.

- The Italian intrigue (Ferro, Tartaglia, Cardano, Ferrari, 1500s)
- Square roots of negative numbers
- The quintic and the legend of Galois (France in the 1800s)