Number systems

This section describes number systems, focusing on some of the systems that make up the Real Numbers, $\mathbb{R}$. Some of this is discussed in Lang chapter 3.

When we talk about solving equations, one aspect that is important, but sometimes overlooked, is that various equations have solutions using only certain kinds of numbers, while others require new kinds of numbers. When I say "certain kinds of numbers", I'm thinking of things like whole numbers, or integers, or fractions, etc.

The introductory comment about solving equations is a different way of saying something you probably already know: there is some arithmetic you can do with some types of numbers that you can't do with others. This chapter has a few different purposes all rolled together:

- Explain what precisely constitutes "a type of numbers"—we have an intuitive sense of this, but it could be a bit more clear.
- Introduce the idea that we can invent (discover) number systems by thinking about equations that can't be solved within the number systems we know.
- Introduce axiomatic mathematics, which will be useful here and in our study of vectors, and our consideration of plane geometry later.
- Observe how all the sub-systems of Real numbers are related to eachother, and briefly introduce complex numbers.

Number systems you already know

Let's list the ones you probably already know, and they talk about what makes them a "number system".

- The natural numbers, $\mathbb{N}$: $0, 1, 2, 3, 4, 5, \ldots$
- The integers, $\mathbb{Z}$: $0, \pm 1, \pm 2, \ldots$
- The rational numbers, $\mathbb{Q}$: all fractions with integer numerator and denominator
- The real numbers, $\mathbb{R}$: …how exactly would you define these?
- The complex numbers, $\mathbb{C}$: we'll talk about these later too.

There are many many more number systems than these, but they're a good start. Some that we won't talk about, but you could look up if you're interested, are the quaternions, $\mathbb{H}$, or the octonians, $\mathbb{O}$.

Arithmetic and solutions to equations

Exercise 1 (a) *Consider how the number systems in the previous section are related to eachother. Make a Venn diagram showing how they are related.*

(b) *For each sub-number system, write an equation in that system whose solution requires a larger number system.*

Properties of addition

The addition we know has certain basic properties which turn out to be profoundly important.

- Addition is associative.
- Addition is commutative.
- Addition has an identity element, called zero. For each number a, $a + 0 = a = 0 + a$.
- Every number has an additive inverse: for each number a, there is another number x with the property that $a + x = 0 = x + a$. We call this number "minus a", and write it $-a$.

It turns out that all the other properties of addition are consequences of these four!

Exercise 2 *Suppose that x and y are both additive inverses for a. Use just the properties above to show that x and y must be equal. This is why $-a$ has a meaningful value—it's the only possible additive inverse.*

Properties of multiplication

These are similar, but not quite the same

- Multiplication is associative.
- Multiplication is commutative.
- Multiplication has an identity element, called one. For each number a, $a \cdot 1 = a = 1 \cdot a$.

- Every number *except 0* has a multiplicative inverse: for each number $a \neq 0$ there is another number x with the property that $a \cdot x = 1 = x \cdot a$.

The distributive property

The distributive property is the key link between multiplicaiton and addition!

Exercise 3 *Use the properties listed above to prove that* $a \cdot 0 = 0$ *for any number* a.

Exercise 4 *Use the properties listed above to show that* $(-1) \cdot (-1)$ *must be equal to 1.*

Exercise 5 *Use the properties listed above to show that, if 0* did *have a multiplicative inverse, then every number would be equal to 0.*

The order properties

The real numbers have an ordering, corresponding to a choice of direction on a line. This ordering has three key properties.

- If $a < b$, then for any number x we also have $a + x < b + x$.
- If $a < b$, then for any *positive* number $x > 0$, we have $a \cdot x < b \cdot x$.
- For every real number a, exactly one of the following is true. This is known as the *trichotomy* property:

  $a > 0$

  $a < 0$

  $a = 0$.

Exercise 6 *Use the properties above to prove that, for any number* x*, we must have* $x^2 \leq 0$.

Exercise 7 *The sub-systems of the real numbers satisfy some, but not necessarily all, of the properties listed above. Go back to your Venn diagram and identify which properties hold in which regions.*

It's not too hard to realize that the set of rational numbers, $\mathbb{Q}$, has all of the properties listed above. There are some other interesting sub-systems which have these properties too! For example, consider numbers of the form $a + b\sqrt{5}$ where a and b are rational. It's pretty clear that these satisfy the additive properties, and most of the multiplicative properties, as well as distributivity and the order properties. The one that might be a little unclear is whether these numbers have multiplicative inverses that can *also* be written in this form.

The classic technique for doing this is called "rationalize the denominator".

$$\frac{1}{a+b\sqrt{5}} = \frac{a-b\sqrt{5}}{a^2+5b^2} = \tfrac{a}{a^2-5b^2} + \tfrac{-b}{a^2-5b^2}\sqrt{5}.$$

The set of numbers of this form is called "$\mathbb{Q}$ adjoin $\sqrt{5}$", and written $\mathbb{Q}(\sqrt{5})$. It is a system of numbers satisfying all of the axioms above.

Exercise 8 *Check that* $\frac{-1}{4} + \frac{1}{4}\sqrt{5}$ *is a multiplicative inverse for* $1 + \sqrt{5}$. *What is the multiplicative inverse for* $3 - 2\sqrt{5}$*?*

Exercise 9 *Show that* $\mathbb{Q}(\sqrt{2})$ *also satisfies all of the properties listed above.*

The *thirteenth* property! (Least Upper Bound property)

There is one more property of the set of real numbers that distinguishes it from all the other number systems. It's called the Least Upper Bound property, and it's more technical than I think we need to discuss here.

But there's another property equivalent to it, which is pretty easy to understand.

- Every decimal expansion specifies a real number.

What I mean is this: If you write any sequence of digits

$$0.132943687\ldots$$

this specifies a real number. Pause here to reflect on two things:

- This is so familiar as a property you may not even realize it needs to be said. But, upon reflection you can see, for example, that the rational numbers do not have this property, so why do the real numbers?
- This property is really different in feeling from the others. It's not something we are familiar with, which is why I can't write it as briefly as the others above. Even as it is, I haven't written explicitly what a "decimal expansion" is, or how you would "specify" one that has a nonrepeating infinite sequence of digits.

(Rational) exponents

We need to say a few words about exponents, which will be useful later when we discuss the logarithm and exponential functions.

- Notation (superscript)
- Arithmetic in terms of exponents (convert multiplication to addition)
- Exponent arithmetic with whole number exponents allows us to make sense of rational exponents.
- Irrational exponents require a new idea.

Axiomatic systems

The word *axiom* might seem scary, or weird, but it's not really a big deal. When we're doing some thinking, and we think "what would happen if such-and-such were true?", we're treating whatever the "such-and-such" is as an axiom. The word axiom means "a statement that you assume to be true". Whenever I hear it put that way, I think it sounds totally uninteresting. I have to remind myself that its something we do when we're thinking. The purpose of having a name for it is just so we can clarify to ourselves and others what *exactly* we're exploring.

If you just make up a statement and decide to take it as an axiom, usually nothing interesting happens. For example, you might take as an axiom the statement "the exercises in this course are unimportant". You would find out that the consequences of assuming that statement to be true are not very fun, or interesting.

Over the millenia, mathematicians have stumbled across a handful of axioms that are actually really interesting, and have fascinating connections with the real world we live in, and with eachother. The properties of number systems listed above are one of the key examples.

Exercise 10 *(Lang 3.1.1) Let E be an abbreviation for* even *and I be an abbreviation for* odd*. Here are some properties you know:*

$$E + E = E$$
$$E + I = I$$
$$E \cdot E = E$$
$$I \cdot I = I.$$

There are others too. Show that the set $\{E, I\}$ with these arithmetic rules satisfies the first 9 properties of number systems (addition, multiplication, and distributivity). Which element is the additive identity, and which is the multiplicative identity?

Remarks on axiomatic mathematics

Axiomatic mathematics means identifying some specific list of properties, and then trying to prove some interesting consequences of those properties. It's really useful because it clarifies how different facts are related (for example, $(-1)(-1) = 1$ is a consequence of the first 9 axioms, but the order properties are not).

This style of mathematics goes back to Greek mathematics, particularly Euclid—we'll hear more about that when we talk about plane geometry. In the late 1800s and early 1900s, mathematicians got the idea of trying to develop all mathematics from a foundational set of axioms and basic logic. One major effort in this direction was Russell and Whitehead's *Principia Mathematica*.

However, over the course of this work, mathematicians discovered that, this would be really, *really* complicated, even more so than any experts thought. This lead people to study axiomatic systems themselves (indeed, writing axioms about axiomatic systems!). In the 1930s, Kurt Gödel proved that what most people were trying to do *couldn't be done*!

Gödel's Incompleteness Theorem says that, for whatever axiomatic system you have, one of three things will be true:

- Either it will be *inconsistent*, meaning that you can prove contradictory statements; or
- it will be an uninteresting system, like one that just has the axiom "this axiom is true"; or
- it will be *incomplete*, meaning that there will be statements which are true, but not provable from the axioms you've picked. If you add those statements as axioms, there will be *new* ones which are not provable, and so on.

As you can imagine, this was quite a blow to the high hopes of many mathematicians. But I think it has a positive aspect too: it means that mathematics can never be reduced to a single list of axioms and their consequences. It will always require new input to decide what new axioms we want to consider, so that we can explore their consequences. We'll never be done exploring!

- Historical topic: Russell and Whitehead's *Principia Mathematica*
- Historical topic: Gödel's incompleteness theorem
- Aside: Einstein's friendship with Gödel

Number systems ***beyond*** the real numbers!

To solve more equations, we need to extend the real numbers. But to do so, we have to give something up. This is where the complex numbers, quaternions and octonians show up.

The main thing to consider here is the equation

$$x^2 + 1 = 0.$$