Vector arithmetic

Arithmetic for numbers can be extended to arithmetic for points in the plane, or in higher dimensions.

Vector arithmetic

We understand how to identify points on a line with elements of the real numbers, $\mathbb{R}$. The point corresponding to the number 0 is called the "origin", and each other number x corresponds to a point distance x from 0; to the right by $|x|$ if $x > 0$, and left by $|x|$ if $x < 0$.

The addition/subtraction and multiplication/division for real numbers corresponds to translation and dilation of the real line.

For each natural number $n > 0$, we let $\mathbb{R}^n$ denote the set of n-tuples of real numbers, $(x_1, \ldots, x_n)$. The most familiar cases are the line, $\mathbb{R}^1$, and the plane, $\mathbb{R}^2$. The points of 3-dimensional space are $\mathbb{R}^3$. In our discussion below, we'll focus on the case $n = 2$, but you will see that the same ideas apply for any $n > 0$.

Points in the plane $\mathbb{R}^2$ correspond to pairs of numbers, (x, y). The addition/-subtraction of numbers extends to arithmetic for pairs in what is called *componentwise* arithmetic: For any points (x_1, y_1) and (x_2, y_2), we define addition and subtraction as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

Exercise 1 *Check that this definition of addition for $\mathbb{R}^2$ satisfies the additivity axioms. What are the additive identity and inverses?*

We can also define multiplication by a real number r as follows

$$r \cdot (x, y) = (rx, ry).$$

This is called *scalar multiplication*. Note that this does not give us a way of multiplying points in the plane, but merely a way of multiplying single numbers by points.

There is another operation for points in the plane called the *dot product* or *inner product*:

$$(x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1, y_2.$$

Notice that, although this is commutative and associative, it produces a single number, not a point in the plane. Observe that general linear equations can be expressed using the dot product. The equation

$$2x + 3y = 1$$

is the same as the equation

$$(2,3) \cdot (x,y) = 1.$$

If there were a "division" that reversed the dot product, then we could use it to solve these equations. Of course there is no such division, but we will see, when we talk about matrices, how this can work for systems of equations.

Exercise 2 *The points of $\mathbb{Q}(\sqrt{5})$ can be identified with a rational plane $\mathbb{Q}^2$ as $a + b\sqrt{5} \leftrightarrow (a,b)$. Observe that the addition in $\mathbb{Q}(\sqrt{5})$ corresponds to componentwise addition in $\mathbb{Q}^2$. Observe, moreover, that the multiplication in $\mathbb{Q}(\sqrt{5})$ does give an interesting multiplication to the rational plane. How would things be different if we had started with $\mathbb{Q}(\sqrt{3})$ instead of $\mathbb{Q}(\sqrt{5})$?*

Exercise 3 *The complex number system is $\mathbb{C} = \mathbb{R}(\sqrt{-1})$. Check that this number system satisfies all of the first nine axioms (addition, multiplication, and distributive rule). Is it possible to define a notion of order (positivity) so that $\mathbb{C}$ satisfies the order axioms too?*

Translation and dialation

Now we want to explain the geometric effect of addition and scalar multiplication on the plane.

Exercise 4 *Explain how addition corresponds to translation, and scalar multiplication corresponds to dilation (scaling).*

Exercise 5 *Given two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, explain how the four points $0 = (0,0)$, A, B, and $A + B$ form a parallelogram.*

The distance formula

Explain the Pythagorean theorem and distance. For a point $A = (x_A, y_A)$, we define

$$|A| = \sqrt{x_A^2 + y_A^2}.$$

Equivalently, it is often useful to write

$$|A|^2 = x_A^2 + y_A^2.$$

Exercise 6 *Show that, for a number r and a point A, we have $|r \cdot A| = |r| \cdot |A|$.*

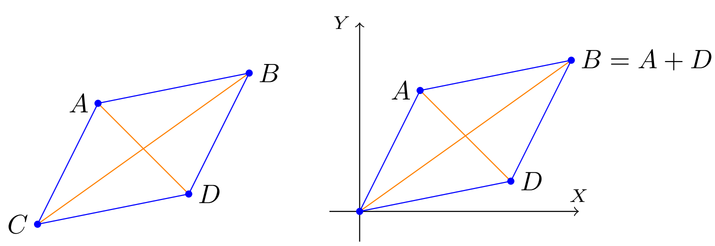
Exercise 7 *Write a formula for the distance between two points A and B. Compare this with the definition above for $|B - A|$.*

Exercise 8 *Use the distance formula to write an equation for the set of all points which are distance 3 from the point $A = (-2, 2)$.*

The Parallelogram Law

The Parallelogram Law states that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides. It is a generalization of the *Pythagorean Theorem.*

We will give a proof. Consider a parallelogram as shown below. We can add coordinates with one corner at the origin, also below.



The Parallelogram Law: $2\,AB^2 + 2\,AC^2 = AD^2 + BC^2$

Let's let (x, y) be the coordinates of A and (u, v) be the coordinates of D. Because we have a parallelogram, the coordinates of B are $A + D = (x + u, y + v)$.

Now we can express what we're trying to prove using coordinate arithmetic:

$$
\begin{aligned}
AB^2 &= |B - A|^2 = |D|^2 = u^2 + v^2 \\
AC^2 &= |A - 0|^2 = x^2 + y^2 \\
AD^2 &= |D - A|^2 = (u - x)^2 + (v - y)^2 \\
BC^2 &= |B - 0|^2 = (x + u)^2 + (y + v)^2.
\end{aligned}
$$

Exercise 9 *Expand the expressions for AD^2 and BC^2. Then use these to verify the Parallelogram Law.*