Constructing Spectral Sequences

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http://www.nilesjohnson.net/ss-construction.html

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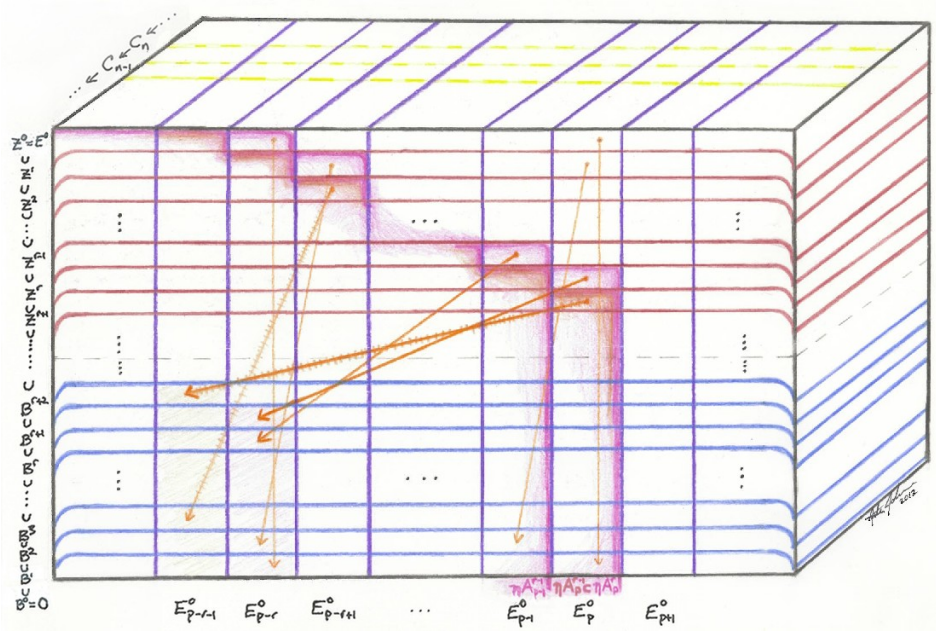


We give a schematic diagram for the construction of a spectral sequence for the homology of a filtered complex. These slides assume that the reader is familiar with the basic structure of spectral sequences. Our goal is to give a visual aid to complement the standard construction. Our specific notation follows §5.4 of

Weibel, *An introduction to homological algebra*. Cambridge studies in advanced mathematics, **38**. 1994.

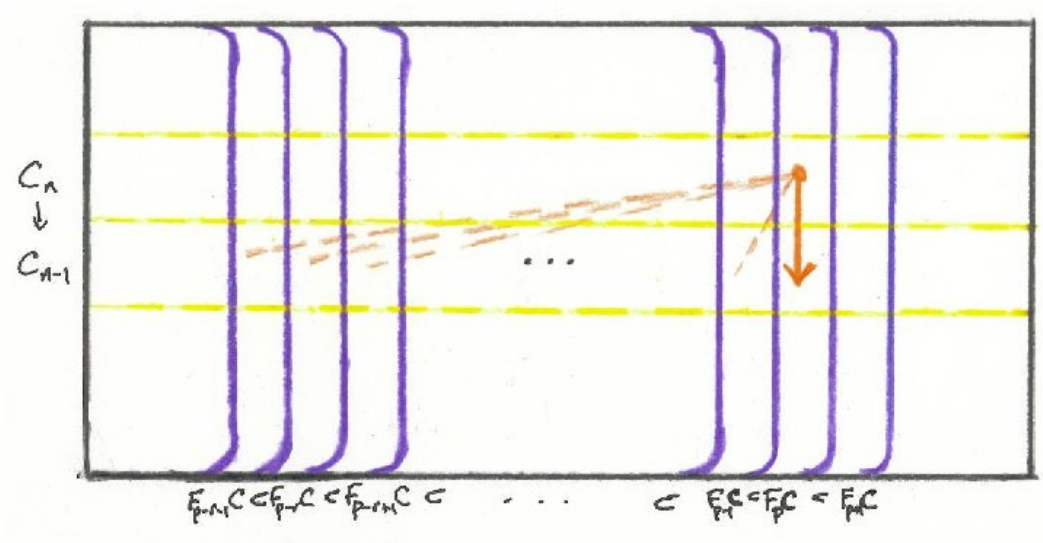
As Weibel does, we drop the complementary degree q for readability.

Let C be a filtered chain complex. For each n:

$$\cdots \subset F_{p-1}C_n \subset F_pC_n \subset F_{p+1}C_n \cdots$$

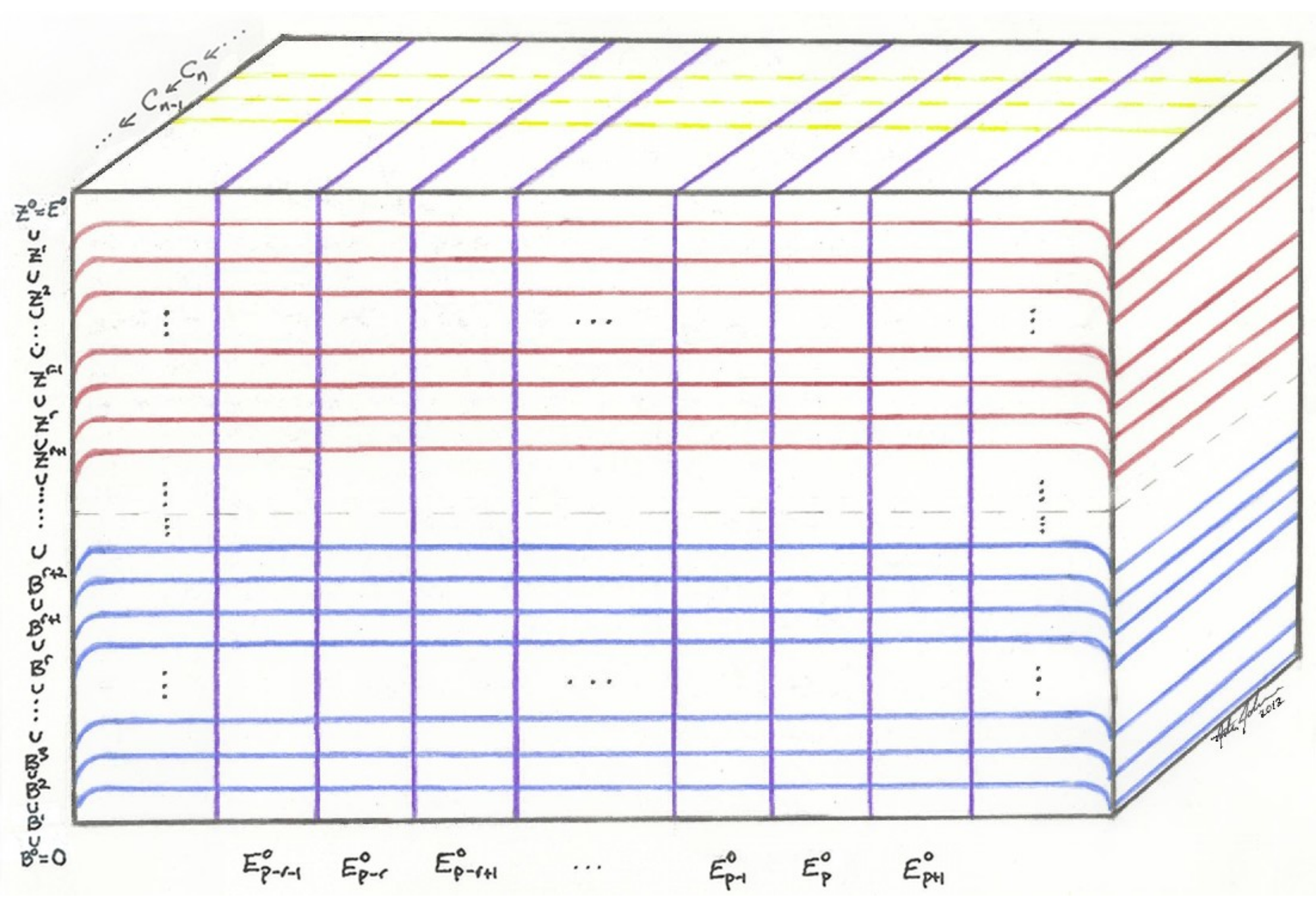
$$\eta_p : F_pC \to F_pC/F_{p-1}C$$

$$A_p^r := \left\{ c \in F_pC \,\middle|\, d(c) \in F_{p-r}C \right\}$$



$$Z_p^r := \eta_p A_p^r$$

$$B_p^r := \eta_p dA_{p+r-1}^{r-1}, \quad B_{p-r}^{r+1} := \eta_{p-r} dA_p^r, \quad B_{p-r-1}^{r+2} := \eta_{p-r-1} dA_p^{r-1}$$


$\cdots \leftarrow C_{n-1} \leftarrow C_n \leftarrow \cdots$
$Z_1^0 = E^0$
$C \cdots C Z_2^1 C Z_1^1 C$
$C \cdots C Z_{r+1}^r C Z_r^r C$
$C \cdots C Z_{r+1}^{r+1} C$
$C \cdots C B_{r+1}^{r+1} C B_r^{r+1} C$
$C \cdots C B_3^3 C B_2^2 C B_1^1 C$
$B_0^0 = 0$
E_{p-r-1}^0
E_{p-r}^0
E_{p-r+1}^0
$\cdots$
E_{p-1}^0
E_p^0
E_{p+1}^0
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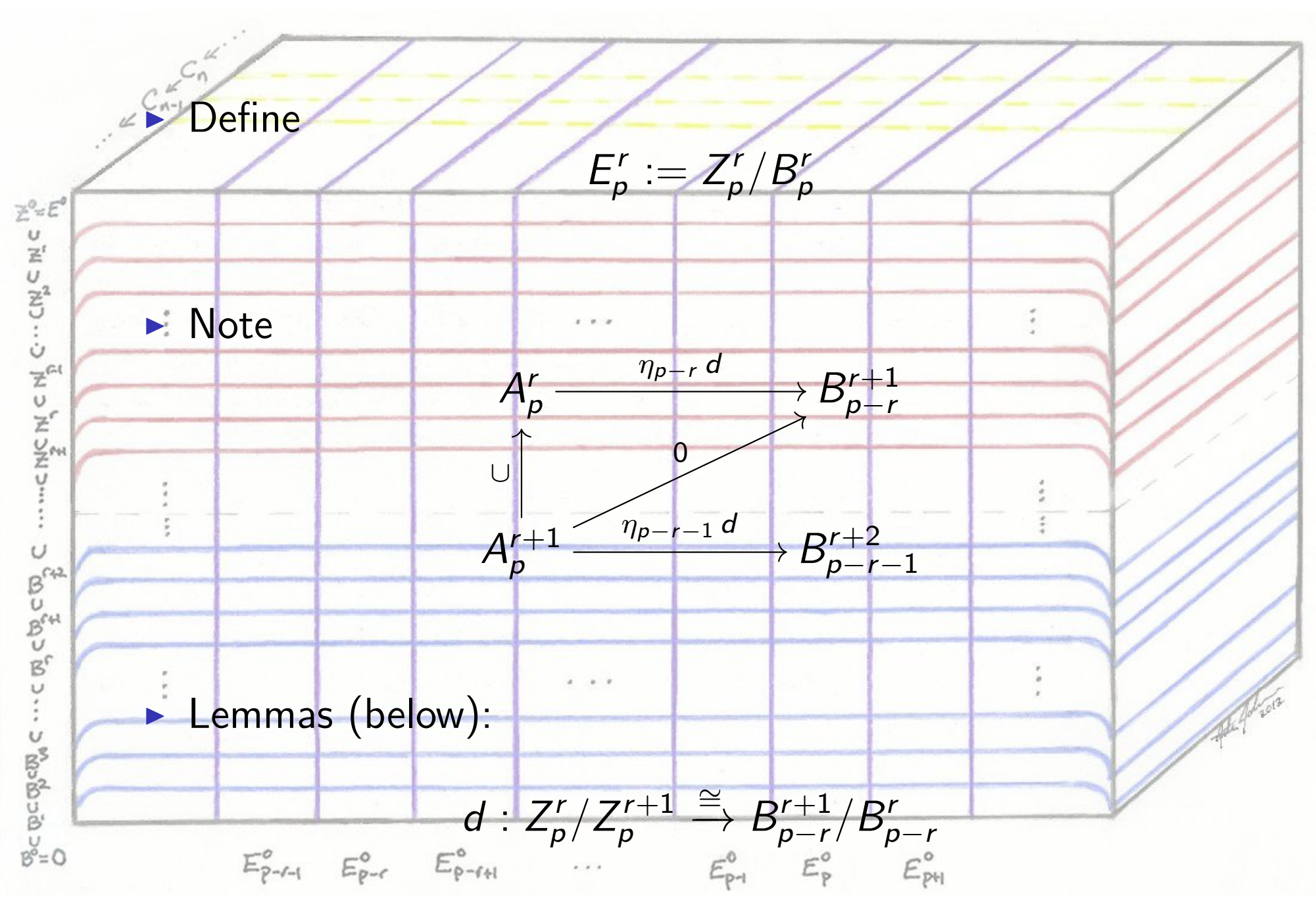
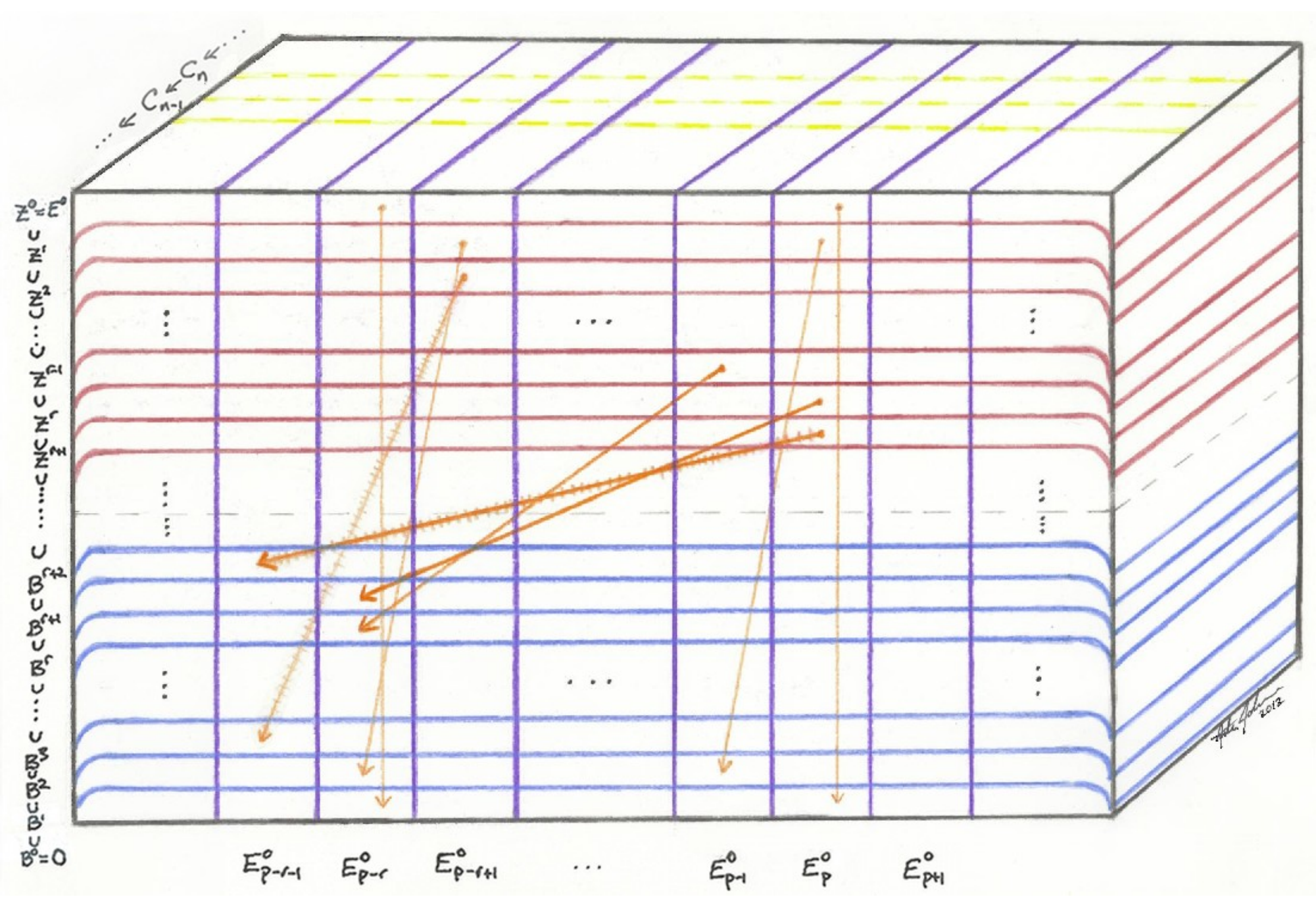

Define
$E_p^r := Z_p^r / B_p^r$
Note
A_p^r
$\eta_{p-r}\, d$
B_{p-r}^{r+1}
$\cup$
0
A_p^{r+1}
$\eta_{p-r-1}\, d$
B_{p-r-1}^{r+2}
Lemmas (below):
$d : Z_p^r / Z_p^{r+1} \xrightarrow{\cong} B_{p-r}^{r+1} / B_{p-r}^r$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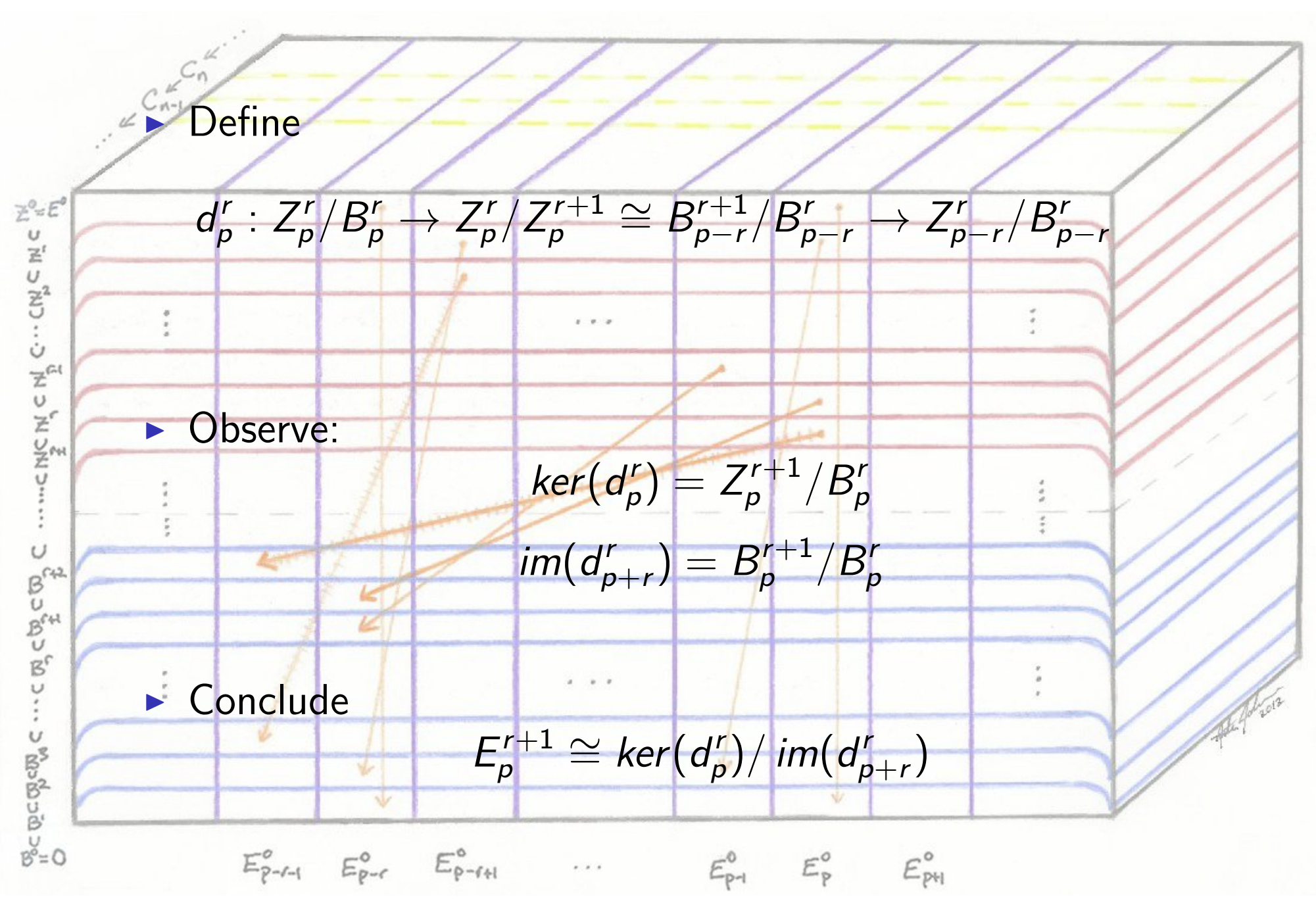
$C \cdots C Z_{r+1}^{r-1} C Z_r^r C$
$C \cdots C B_r^r C B_{r+1}^r C B_{r+2}^r C$
$B_0^0 C B_1^1 C B_2^2 C B_3^3 C \cdots C B_r^r C$
$B^0 = 0$
E_{p-r-1}^0
E_{p-r}^0
E_{p-r+1}^0
$\cdots$
E_{p-1}^0
E_p^0
E_{p+1}^0



- Define

$$d_p^r : Z_p^r/B_p^r \to Z_p^r/Z_p^{r+1} \cong B_{p-r}^{r+1}/B_{p-r}^r \to Z_{p-r}^r/B_{p-r}^r$$

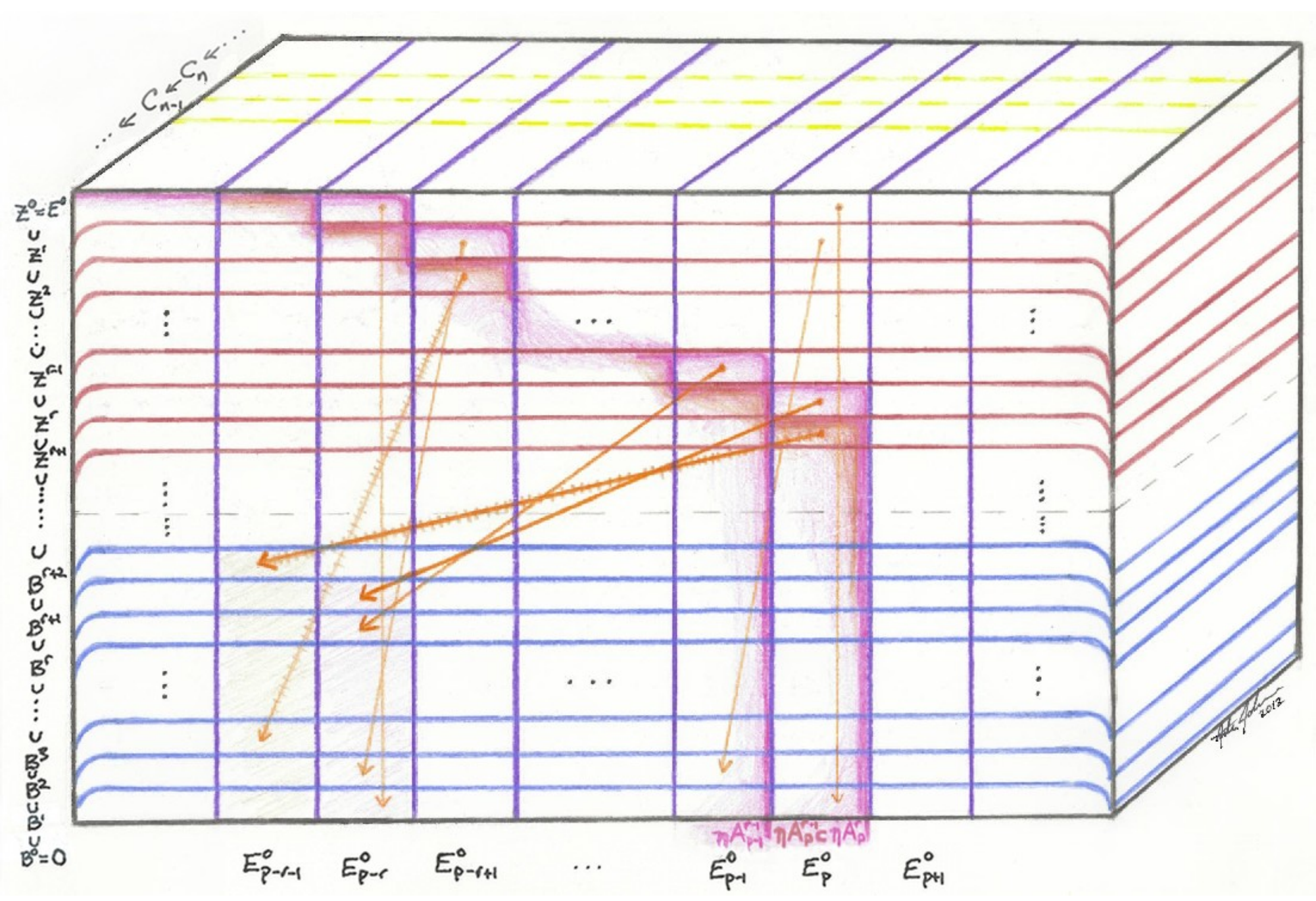
- Observe:

$$ker(d_p^r) = Z_p^{r+1}/B_p^r$$

$$im(d_{p+r}^r) = B_p^{r+1}/B_p^r$$

- Conclude

$$E_p^{r+1} \cong ker(d_p^r)/\, im(d_{p+r}^r)$$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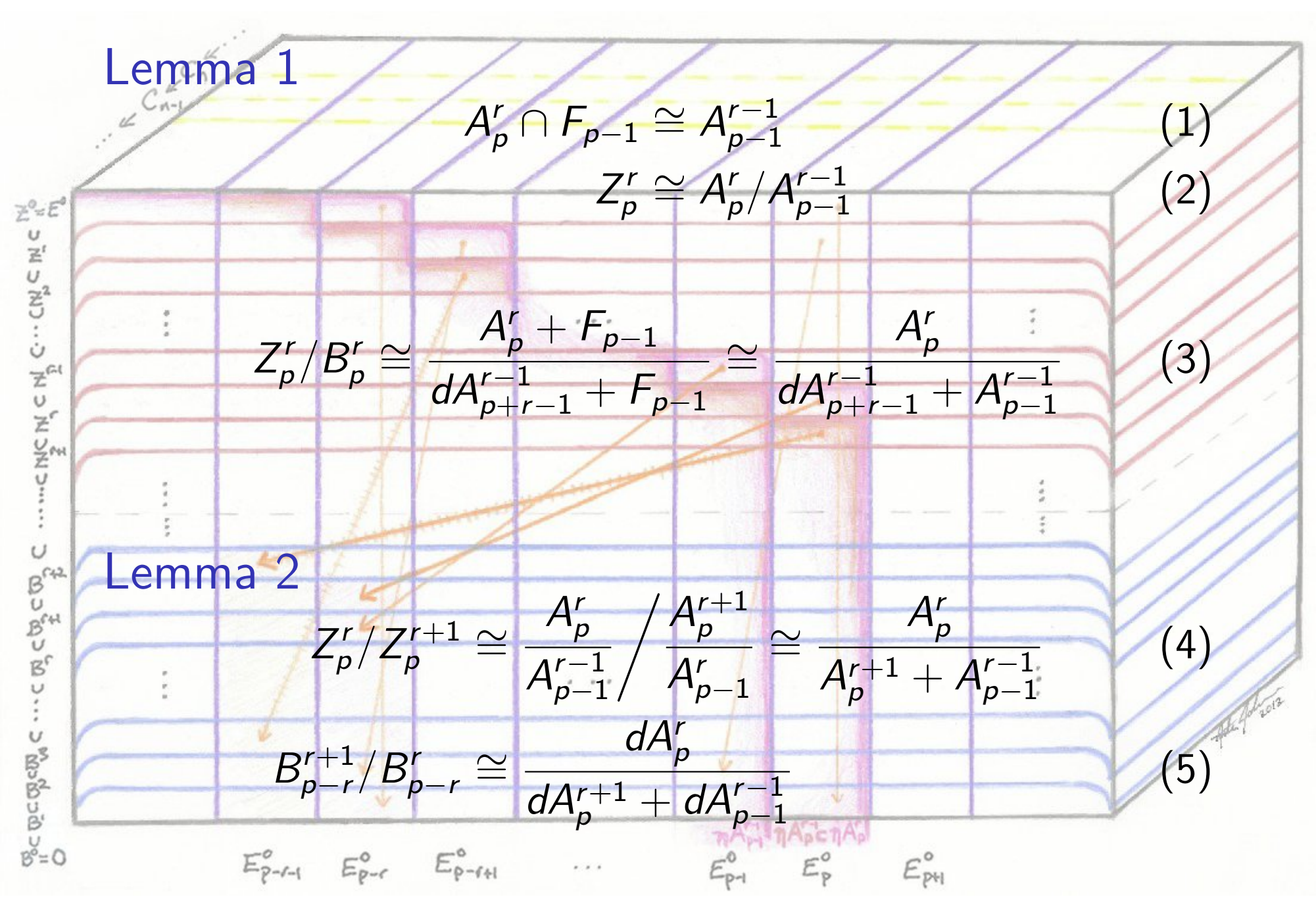
$C \cdots C B_3^1 C B_2^1 C B_1^1 C$
$B^0 = 0$
E_{p-r-1}^0
E_{p-r}^0
E_{p-r+1}^0
$\cdots$
E_{p-1}^0
E_p^0
E_{p+1}^0
ηA_{p-1}^{r-1}
ηA_p^{r-1}
ηA_p^r



Lemma 1

$$A_p^r \cap F_{p-1} \cong A_{p-1}^{r-1} \tag{1}$$

$$Z_p^r \cong A_p^r / A_{p-1}^{r-1} \tag{2}$$

$$Z_p^r / B_p^r \cong \frac{A_p^r + F_{p-1}}{dA_{p+r-1}^{r-1} + F_{p-1}} \cong \frac{A_p^r}{dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}} \tag{3}$$

Lemma 2

$$Z_p^r / Z_p^{r+1} \cong \frac{A_p^r}{A_{p-1}^{r-1}} \Big/ \frac{A_p^{r+1}}{A_{p-1}^r} \cong \frac{A_p^r}{A_p^{r+1} + A_{p-1}^{r-1}} \tag{4}$$

$$B_{p-r}^{r+1} / B_{p-r}^r \cong \frac{dA_p^r}{dA_p^{r+1} + dA_{p-1}^{r-1}} \tag{5}$$